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THE MATHEMATICS TEACHER

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Editor and Chairman of the Committee on Official Journal—E. H. C. HILDEBRANDT, 212 Lunt Building, Northwestern University, Evanston, Illinois.

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THE MATHEMATICS TEACHER

Volume XLV Number 1

What Mathematics Shall We Teach in the Fourth Year of High School?*

By C. C. MacDuffee
University of Wisconsin, Madison, Wisconsin

I THINK I might as well start right out with a statement of what I think should be the content of the fourth year of mathematics in high school. I think it should be a course in which advanced algebra and the rudiments of analytic geometry are integrated, and which contains a few of the essential ideas of the calculus in the second semester.

Nou are probably saying to yourselves that this is not a new suggestion, and I know that it is not. It has been and is being offered in many of the forward-looking schools of the nation. It has been offered in several of the schools of New York City. It is now being offered in several of the schools in Wisconsin, including the high school administered by the Department of Education of the University of Wisconsin. Hence it is not an untried experiment. But it is far from being standard procedure in our schools.

Although our program this morning is supposed to be concerned with gifted students, the proposed curriculum need not be restricted to the gifted, except in the sense that mentally inferior students seldom elect four years of mathematics. It is not a harder program than the usual one, it does not require a higher order of ability or longer lessons.

Two weeks ago one of my better students in a course which I am giving in the Theory of Equations, a university junior, was in my office. I took the opportunity to ask him about his high school mathematics, being careful not to prompt him in any way. He had had four years of mathematics in high school, he stated. I asked him about the content of the fourth year course.

"Well, to be honest," he said, "it was just a repetition of what we had had before."

"Did you find this interesting?" I asked.

The reply was, "No."

Now this situation is not universal, but it is more common than it should be. Students sometimes get the idea that by the third year in high school they have learned about all the mathematics there is and that the fourth year is just a review with problems that are a little harder and a little longer. This is not a stimulating point of view.

I believe that analytic geometry can be used to illustrate and motivate a course in algebra if the analytics is given in small doses well diluted with algebra. The theory of straight lines and planes is the best possible illustration of linear systems of equations and determinants. Circles and

* An address delivered at the Twenty-Ninth Annual Meeting of the National Council of Teachers of Mathematics at Pittsburgh, Pennsylvania, March 31, 1951.

other conics provide laboratory material for all properties of quadratics which are usually considered. Accompanied by geometric illustrations, the abstract concepts of algebra become alive and obvious.

This course in analytics should consist of more than the mere plotting of curves. Much curve plotting is now being done in fourth year mathematics courses and a little of it is of course desirable. But there is no reason to shy away from the theorems and proofs of analytic geometry. It is to the theoretical side of the subject that the students' minds must be adjusted if the transition from high school to college mathematics is to be made easier.

The amount of calculus to be introduced in the second semester would be minute. It would consist merely in finding tangents to particular conics, and perhaps a few areas, problems which date back to the pre-Newtonian period in mathematics, but problems which prepared the way for the calculus.

At the present time many students find the first year of college mathematics quite difficult. The pace in college is faster than in high school, and the standards are frequently higher. There is no time to go over a topic a second time. These differences in teaching technique sometimes disconcert a freshman. A coed at Wisconsin recently reported to her adviser that her mathematics instructor was a poor teacher. "Why," she said, "as soon as the class gets so we can do one thing, he goes right on to another."

I have a great deal of sympathy for the student who meets analytic geometry for the first time. I recall that I failed my first test in the subject. The point of view is so different that many students find that it takes time to acquire it. If the student could begin his college course with some understanding of this point of view, he could immediately begin to get something out of his mathematics. Even if he had failed the course in high school, he would probably have gotten enough to achieve this objective.

A year used to be the time allotted to analytic geometry, but now many colleges are trying to crowd it into a semester or even less. On looking back over the subject, it seems easy, but not so to the poor student meeting it for the first time.

After all there is but a year's difference between the high school senior and the college freshman. If it is possible to give a condensed and rapid course in analytics to a not-too-well prepared freshman, it is certainly possible to give a leisurely course in the same subject to a high school senior.

I think that the colleges and the high schools should work together in trying to ease the jolt which many students receive in passing from the high school to the university. One of the worst of these jolts occurs in the field of mathematics and the course which I am advocating is certainly an excellent shock absorber. This is not mere theory, it is a proven fact.

All scientists and engineers, and most persons who desire to understand the modern world, need to take a course in physics. The ideal place for this first course in physics is the freshman year. Now there are two ways of teaching physics, without the calculus and with the calculus. The first way is superficial, full of rationalizations and obscure places. The second way is clear, simple and satisfactory. But very few colleges teach elementary physics the second way, because their students have no concept of the calculus.

Now the amount of calculus needed to understand elementary physics is very slight. Only the concepts of derivative and integral of the simplest functions are usually needed. In fact, if the student even knows what the concepts mean, the physics instructor can build from there. This problem is not one which I have invented to bolster the cause of analytics and calculus in the high schools. It is a serious problem in all engineering schools, and was, in fact, the problem which first aroused my interest in this project. If every student entering the college of

engineering had even a rudimentary acquaintance with analytics and the calculus, several serious problems now bothering the engineering colleges would be solved.

The schools of many of the nations of Europe differ from American schools in a number of ways. One is that the European schools are concerned with the education of leaders, while American schools are at the present time striving to produce some imaginary individual known as the ideal citizen. No one has ever seen this ideal citizen, no two educators agree on what he looks like or what qualities he should possess. In 1951 he must be well adjusted to his environment. Next year the slogan will be different, but as far as I can see the boy remains about the same from year to year.

But let us return to the consideration of the European schools, and restrict attention to their curricula in mathematics. Although the standard of performance is higher than in our schools, their curricula are not as different from ours as one might expect. In fact, the usual courses in arithmetic, algebra, plane and solid geometry, and trigonometry are not very different from ours. It is mainly in the eleventh and twelfth years, particularly in the twelfth year, that they differ materially from courses in our schools.

My knowledge is not at first hand, for I have never attended European schools, but I have talked with persons who have either attended or been officially connected with the schools of England, France, Switzerland and Germany. In all of these schools modern concepts and methods are employed in geometry, and the algebra, geometry, trigonometry and mechanics are more integrated, each being used to illustrate and support the others. In the twelfth year a solid course in the calculus is given. This is beyond what I am willing to advocate for American schools at the present time.

Why is it that our high school mathematics program seems to have run out of gas in its third year? Why does the program that has kept pace with European schools for eleven years flatten out in the twelfth year? I do not pretend to know the complete answer.

But one cause is the conservatism of our schools. We can call our schools progressive and boast of our stream-lined atomic-age buildings with built-in television, but actually it is difficult for a new idea to get into a mathematical text book. One elementary text-book writer copies from some previous elementary text book. College text books have surprisingly little influence on high school texts in the same subject, but then again college text-book writers are not immune from the same disease. Incorrect and out-moded ideas are perpetuated from generation to generation. Perhaps mathematics suffers peculiarly from this conservative attitude, but then again perhaps we are not in a position to notice it so much in other subjects.

But I believe that the main block in the path of teaching more advanced mathematics in the high schools is other than conservatism. I have been warned that this is a dangerous subject and that I should approach it cautiously. But I am noted as a bull in a china closet, so I will risk breaking a few teacups, and perhaps losing a horn or two myself in the discussion which is to follow. I believe that the main reason why many schools are not more favorable to the teaching of analytic geometry and the calculus is that they do not have teachers who feel capable of teaching it.

This criticism cannot be leveled at all schools or at all teachers. Pardon me if again I use as illustration the state which I know best. Each year the University of Wisconsin turns out about a dozen men and women who are given degrees in Education with major in mathematics, and a teacher's certificate. These persons have had a year of calculus, a semester each of the Theory of Equations, College Geometry, Projective Geometry, and Differential Equations, or an equivalent program, and usually a year of physics.

Most other universities have similar courses, and thus each year a small number of well-trained teachers of mathematics are produced. These teachers are quite capable of handling a high school course in the calculus, and would doubtless be eager to do it.

There are many institutions in the United States which are producing teachers of mathematics, and their standards of competence are extremely variable. These standards do not depend upon the size or location of the institution, for many of the best colleges are small, while some of the largest are so concerned with teaching others how to teach that they flagrantly overlook the question of whether their graduates know anything well enough to teach it.

But the most discouraging feature of the whole situation is the number of active teachers who do not pretend to competence in their fields. Most teachers teach in more than one subject, and they cannot be expected to have majored in more than one field. Indeed, they may not even have a minor in the subject they teach. I recall having had to teach a class in German once as a substitute, but only for a very short time, I assure you.

A couple of years ago I talked with a man from the University of Minnesota who had studied this problem in his own state. He had reached the conclusion that approximately half the high school teachers of mathematics in Minnesota were certified in that subject. Others were football coaches, social science teachers and the like who had been given a class in mathematics on an emergency basis long ago, and the emergency had become permanent. I have no reason to believe that the situation in Wisconsin is materially different. It is fairly easy to believe that teachers not well prepared in their subject are not eager to introduce more advanced courses.

Another road block is the attitude of many principals, superintendents and vocational guidance directors toward mathematics. Most of them seem to regard it as Calvin Coolidge's minister regarded sin. They are against it. It makes trouble for many students, these students are unhappy, their parents are unhappy, and consequently the principal is unhappy. Therefore, let us all be happy and throw out mathematics. It is the philosophy of the proverbial ostrich. Real ostriches have more sense.

Perhaps I am getting too far from the main purpose of this conference. It is primarily concerned with the education of the gifted, and I wish to say "amen" to everything that Professor Price* has said. But I am assuming a somewhat broader definition of the gifted student than he is. The program which I am advocating is designed not only for the highly gifted but for all students who are capable of entering college and pursuing a course in the natural sciences. I am sure that all such students would find their college work easier, and they would be happier, if they could enter college with a little analytics and calculus. It would not be long until this news would drift back to the high schools, and mathematics would become a more popular subject.

There is an occasional note in modern educational theory that disturbs me greatly. It is a note of defeatism. It states that most of our American students are ineducable and that we should not try to teach them anything more than they need to know to live and eat and reproduce their kind. A distressing article on this theme appeared in the *Saturday Evening Post* a little over a year ago written by a Miss Frances Rummell, who was connected with the office of the United States Commissioner of Education.

Contrary to the opinion voiced in this article, I believe that a large share of our students would profit from a curriculum designed for the moderately gifted. Many of them are by no means as dumb as they

* See G. Baley Price, "A Mathematics Program for the Able," THE MATHEMATICS TEACHER (Oct., 1951), pp. 369–76.

appear. They are not as dumb as they believe themselves to be, their opinion of their own dumbness strengthened as it frequently is by guidance directors, principals and exasperated teachers. If you can convince a boy that he is dumb, he will act dumb, as any competent psychologist will tell you. If you can convince him that he is bright, the chances are that he will strive to prove you correct.

Of course there is the point of view among adolescents that it is smart to be dumb. This is particularly common among boys. This attitude frequently takes the form of considering aptitude for one's studies to be an unmasculine attribute. He-men are primarily concerned with athletics, because, isn't the only man on the faculty the coach? It is by no means uncommon for a bright boy deliberately to conceal his brilliance, even to try to keep from getting as good grades as he could.

It is clear from these considerations that teachers are not always creating the optimum atmosphere for scholarship within their schools. Perhaps it is not the fault of the teachers, for traditions in the home are frequently dismal. But teachers are not always blameless. There are teachers who are not scholars. They were not scholars while they were in college and they have not been subsequently converted. They do not sympathize with the precocious child. In fact, they sometimes resent him and try to dampen his enthusiasm. If allowed to ask questions without restraint, the student could become a source of embarrassment to a teacher who is sometimes just a page ahead of his class.

Besides the mechanical imparting of knowledge, I believe it is the function of the teacher to mold opinion in the school and in the community and to uphold the ideal of scholarship in a world fast becoming superficial. In every school there are a few superior students, necessarily sensitive children because they are superior, who need the guidance of their teachers, who will respond greedily to it, and who will in after years become the pride of the schools that produced them. They are now the forgotten element in the school population in an age when mediocrity is held up as the ideal. Effort spent in the education and guidance of such children will be rewarded a hundredfold.

Never before has mathematics played quite so fundamental a role in civilization as it does now. We are now embarked upon the Age of Science, and modern science is built upon a foundation of mathematics. Every graduate student in chemistry must have calculus, and physicists must go much beyond it. Nuclear physics is guided by abstract mathematical concepts, theory coming before experiment. When the first bomb exploded at Los Alamos, the power of abstract mathematical procedure was dramatically attested.

In view of these undeniable facts, one would reasonably assume that the growing importance of mathematics in our world would be reflected in an intensification of its study in the grade and high schools. Arithmetic, algebra, geometry, and trigonometry are still the basic studies on which the higher varieties have to be built, and very few persons who have mastered them ever express regret over a wasted life. On the other hand, I have heard many a college student mourn because of the short-sightedness of a high school guidance director who told him that mathematics was no longer the vogue.

It is one of the anomalies of our American culture that we as a people do not understand the foundations upon which this culture is built. We talk of mass education and then give our students the shell instead of the substance. We spend hours on end discussing the problems of democracy, problems for which no living man has an answer, at the sacrifice of the precise knowledge which keeps us as a nation among the free peoples of the earth.

The proponents of mathematics in the high schools have not been very vigorous in defending their subject. Perhaps they

(Continued on page 9)

Note on Pythagoras' Theorem

By C. GATTEGNO
Institute of Education, University of London, London, England

IN THE following pages we propose to discuss some of the thoughts on the beautiful theorem which still fascinates millions of people all over the world.

Our suggestions are those of a teacher facing youngsters whose mathematical insight is greatly hampered by the fact that they have to learn things chosen by adults at moments chosen by adults for adult reasons. The teacher who is aware of this difficulty will try to introduce as much variation as he can into the treatment of any question.

In teaching a theorem like that of Pythagoras it is usual to suggest a variety of intuitive proofs until the formal classical proof can be developed. In this note we are undertaking a study of the whole question rather than suggesting how it should be developed if used in a classroom. Although what we say is the outcome of real class teaching, the stress here is on the logical content of the lessons we would give.

In Figure 1, let ABC be the right-angled triangle, $A = 90°$; and a, b, c, the name and length of the sides BC, CA, AB, respectively.

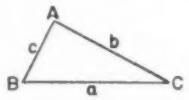


FIG. 1

There are two ways of dealing with the concepts contained in the Pythagorean theorem: areas and similarity. Either we assume areas and assume that we do not know what similarity is, or we start with the concept of similar figures and then apply similarity to problems of areas.

There is obviously the possibility also of starting simultaneously with areas and similarity, but that is somewhat contrary to the deductive and axiomatic tradition prevailing in geometry.

Here we shall separate the two approaches and see what can be done in each case that makes one distinct from the other. It is obvious that experience in teaching will help in the choice of material introduced into actual lessons with secondary school students.

Case I. We assume areas but not similarity.

Let us consider the right triangle ABC and the altitude AH drawn to the hypotenuse (Figure 2). Then Area $\triangle ABH$ + Area

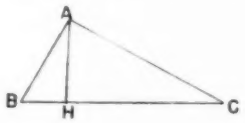


FIG. 2

$\triangle AHC$ = Area $\triangle ABC$. If we unfold the three triangles ABH, AHC, ABC by rotating around AB, AC, BC respectively, we obtain a figure very comparable to the

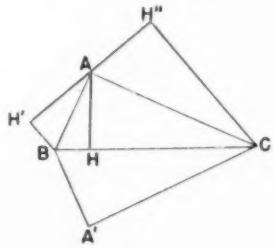


FIG. 3

one in Pythagoras' theorem except that right-angled triangles replace squares. Here we do not need a proof, but we fail to see the relation between this fact and that contained in Pythagoras' theorem.

From this simple result we can deduce an interesting extension of Pythagoras' theorem which appeals to us because it frees us from the static original pattern and allows for the consideration of an infinity of patterns in the same figure.

In Figure 3, let us draw the parallel to

BC at A' and the parallels to AB and AC at H' and H'' respectively (see Figure 4).

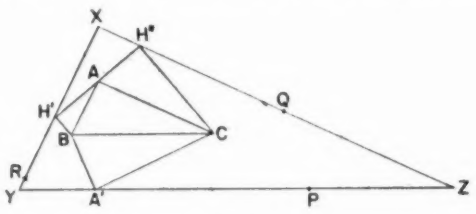


FIG. 4

If P, Q, R are any three points chosen respectively on YZ, ZX, XY we have

Area $\triangle PBC$ = Area $\triangle QCA$ + Area $\triangle RAB$.

And in the case of Pythagoras' theorem

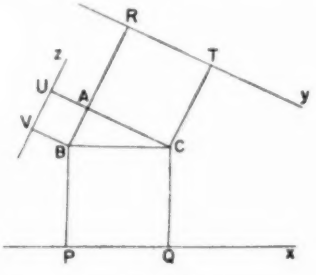


FIG. 5

we get:

"If 1) triangle ABC is such that angle $A = 90°$,

2) the lines x, y, and z are respectively parallel to to a, b and c and their distances are equal to a, b and c respectively,

3) P and Q are two points on x so that $PQ = a$, R and T are two points on y so that $RT = b$ and U and V are two points on z so that $UV = c$; then

Area $PQBC$ = Area $RTCA$ + Area $UVBA$."

How far can one replace right-angled triangles and squares in the proposition in order to obtain extensions, not making use of similarity?

If it is proved that the area of a circle is πR^2, we can give the following figures and results:

In Figure 6, Area C_1 = Area C_2 + Area C_3 since multiplying both sides of $a^2 = b^2 + c^2$ by $\pi/2$ gives the result.

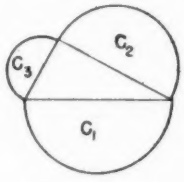


FIG. 6

In Figure 7, Area D_1 = Area D_2 + Area D_3; the factor being $\pi/4$.

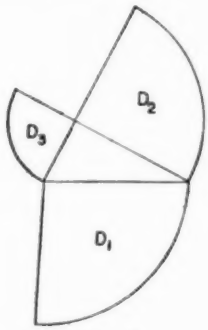


FIG. 7

In Figure 8 Area Z_1 = Area Z_2 + Area Z_3; the factor here being $(\pi/3 - \sqrt{3}/4)$ as can be seen in Figure 9.

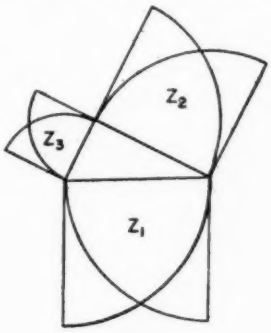


FIG. 8

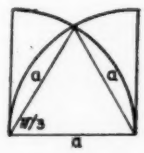


FIG. 9

These extensions are immediate and could be multiplied, but the limitations we encounter are due to the fact that we cannot use any other means of calculation than elementary algebra, and that we cannot make use of similarity.

The following pattern is in the same field of causality and is useful in suggesting some of the important extensions of Pythagoras' theorem.

Let Q_0 be a fixed square, P_0 a point on side BC, P_0x the perpendicular to BC from

P_0. Let P be a moving point on P_0x. In each position we draw the squares Q and Q' on PB and PC. Quite clearly when P is at P_0, $Q+Q'<Q_0$ and when P is at P', $Q+Q'>Q_0$. We can look at the moving pattern and see that by moving P continuously on P_0x, we pass from a sum of two squares less than Q_0 to a sum of two squares greater than Q_0. In the first case the angle at P will be obtuse, in the second case will be acute, and the limit case of the right angle leads to the sum of the squares being equal to Q_0.

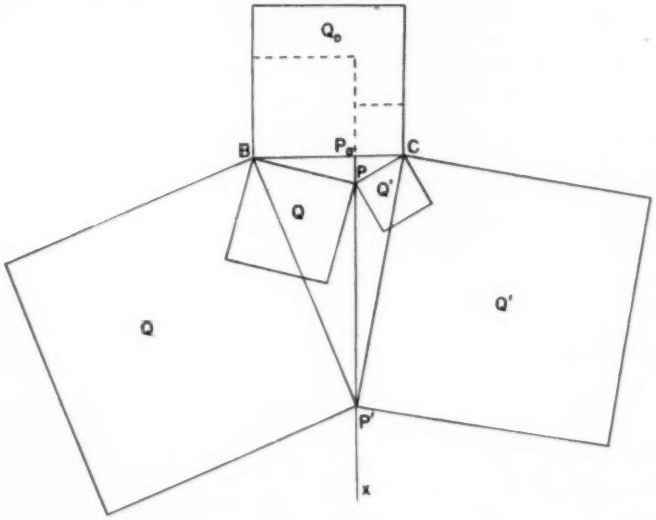


FIG. 10

The consideration of Figure 10 leads therefore to the proposition:

"The sum of the squares on two sides of a triangle is less, equal or greater than the square on the third side according as the angle between these two sides is obtuse, right or acute."

Naturally it is a further step to change the inequalities into equalities by adding or subtracting twice a suitable rectangle, but this pattern has the advantage of putting Pythagoras' theorem in the natural sequence of patterns contained in Figure 10.

Case II. We know similarity and can prove that similar figures have areas in the ratio equal to the square of the ratio of similarity.

From Figure 2 we can show by similar triangles that $a^2=b^2+c^2$, where a^2, b^2, c^2 mean the squares of the lengths of the sides.

"If similar figures are drawn on the three sides of a right-angled triangle then the areas of the figures on the two sides add up to the area of the figure on the hypotenuse."

In fact we have

$$a^2=b^2+c^2$$

and if F_1, F_2, F_3 are the similar figures respectively drawn on a, b, c then

$$\frac{\text{Area } F_1}{\text{Area } F_2}=\left(\frac{a}{b}\right)^2$$

and

$$\frac{\text{Area } F_1}{\text{Area } F_3}=\left(\frac{a}{c}\right)^2.$$

Hence Area F_1 = Area F_2 + Area F_3.

Pythagoras' theorem, as well as the results of Figures 2, 6, 7, and 8, are obviously special cases of this theorem and we see the deep reason why the case of the right-angled triangle (Figure 2) is so like that of Pythagoras, a fact that escaped us in the first instance.

Underlying Pythagoras' theorem are there-

fore the idea of similarity and the obvious result of Figure 2.

Similarity opens the field for an infinity of figures in replacement of the squares of Pythagoras, removing the need for actually finding the factor by which to multiply the equality $a^2=b^2+c^2$, as we had to do previously and this last equality is itself the application of similarity to Figure 2. The fact that area $\triangle ABC$ = area $\triangle ABH$ + area $\triangle ACH$ in Figure 2 is obvious. It does not require any proof and represents the first of a sequence of results of which that of Pythagoras is only one, and indeed no longer such an outstanding one, since squares are not such striking figures when we compare them with other analytic figures.

But we must not forget that the results of Figures 4 and 5 are true even for non-similar figures, and this is an extension obtained within the logical field of the concept of area.

If we add results like those of Figure 10, we see that the two treatments of the question, though they overlap in some respects, are definitely complementary and can bring to our teaching additional sources of light and of understanding.

Our short analysis shows some possibilities of dealing with Pythagoras' theorem in a simple yet quite deep way.

What Mathematics Shall We Teach?

(*Continued from page 5*)

thought their case was so strong that no defense was necessary. This was a perfectly logical point of view, but our age is not a logical one. Self-styled educational experts have stated that mathematics is not important and no one has bothered to reply. People who would like to believe this have, therefore, believed it.

I hope that the time has now arrived when suggestions for the further curtailment of mathematics in our high schools will be met by the counter-claim that not less but much more must be taught. Mathematics is an essential part of our national life as never before. Industry is eager this year to acquire trained mathematicians. More of your Ph.D.'s this year are going into industry than into teaching positions. These will be the men at the top. Below them will be computers and others who have a college degree with a major in mathematics. With men such as these in the key positions, together with similar men from the physical sciences, one can say that mathematics is today an even more vital factor in modern technology than it has been before.

Where does the high school graduate who has not gone to college fit into this picture? I cannot conceive that mathematics will do him any harm. Years ago I worked for a summer at a General Electric Company plant in this state and I recall a young high school graduate in the shops whose ambition had transcended his lack of money to go to college. He studied calculus and electricity all by himself and was moving up the ladder in the employ of the company faster than some of the other young men with engineering degrees.

So, may I ask you to believe in the gifted student, to believe in the importance of scholarship and the subject which you teach, to believe in the future of your country, and to stand fast in your beliefs though the rabble storm without the high school walls.

THIRTIETH ANNUAL MEETING
The National Council of Teachers of Mathematics
Des Moines, Iowa, April 17-19, 1952

Watch for the complete program which will appear in the February issue of THE MATHEMATICS TEACHER. Begin making plans now to attend this important meeting.

Concerning the First Course in Algebra

By HERBERT BERNHARDT
Baltimore, Maryland

IN THE April, 1950 issue of THE MATHEMATICS TEACHER William R. Ransom[1] suggests a more meaningful approach to the teaching of combination of like terms than that usually found in the first course in algebra. The present paper describes an attempt to apply a somewhat similar point of view to the entire beginning course in algebra.

The class began the semester with a complete review of arithmetic, during which they were led to discover the following properties of rational numbers:

1. commutativity: $a+b=b+a$; $a\cdot b=b\cdot a$
2. associativity: $a+(b+c)=(a+b)+c$; $a\cdot(b\cdot c)=(a\cdot b)\cdot c$
3. existence of the identity: $a+0=0+a=a$; $1\cdot a=a\cdot 1=a$
4. function of the zero in multiplication: $0\cdot a=a\cdot 0=0$
5. distributivity of multiplication over addition: $a(b+c)=ab+ac$; $(a+b)c=ac+bc$.

After discussing a number of specific examples the students experienced little difficulty in generalizing their observations into the form expressed above, where a, b, c represent integers or fractions. They were encouraged to investigate how these properties enable one to add a set of numbers in any order and still obtain the same sum, how the algorithm for multiplication is an application of the fifth property listed above, etc. Division was not studied from this point of view because of the time element and because it would not have been germane to the immediate task.

In the second phase of the program the class was asked to consider the four axioms, using examples like:

1. If $6=4+2$, then $6+5=(4+2)+5$
2. If $6=4+2$, then $6\cdot 5=(4+2)\cdot 5$.

Simple number-puzzles were used to introduce equations of the type:

$$8x=12, \quad x+7=10, \quad 6x+8=10.$$

Representation of an unknown number by a letter-symbol was readily accepted by the students because of their previous work in generalizing the postulates of arithmetic. These equations provided little difficulty for most students. Therefore there was no insistence upon the use of the axioms, but the class was shown that in every case the axiomatic method would yield the correct solution. Thus, they gained some appreciation of the axioms but did not form mechanical habits of thought.

Incidentally, equations of the type $8x+4x=20$ led to consideration of "like terms." The problem was clarified by application of the fifth property of rational numbers. That is, $8x+4x=(8+4)x=12x$.

The third stage began after the class had been asked to construct equations to present as challenges to the rest of the class. The equation $x+5=2$ appeared. Discussion of this equation marked the beginning of the investigation of negative numbers. Some children claimed the equation could not be solved because 5 could not be subtracted from 2. A number of students were able to impute meaning to negative numbers. A typical response was: "x equals 3 in the hole." The class agreed to call such a number "negative 3" and to write it as "-3." But when the question was posed concerning the meaning of the product of two negative numbers, there was no continuation of thought.

It was agreed that whatever significance negative numbers might have, the results obtained from elementary arithmetic must

[1] William R. Ransom, "The Apple Theory," THE MATHEMATICS TEACHER, XLIII, No. 4 (April, 1950), pages 172–173.

remain valid. This idea led to a review of the stages of arithmetic progress. First came the counting numbers, then fractions. At this point the definition of fractions as *ratios* of counting numbers was stressed. Starting with this concept the class arrived at the principle: *In no case may fractions be identified with counting numbers.*

The students noted that a special set of fractions seemed to correspond to the counting numbers, namely those with denominator 1. However, referring back to the definition of a fraction as a ratio, they were able to see that even this set of fractions could not be identified with the counting numbers. One result of the discussion was the realization that the rationals constitute an extension of the number concept beyond that of the counting numbers. One student expressed this idea by showing that by accepting the new number concept it became possible to think of division as applicable to all numbers. This was obviously not the case in the counting-number system. Without trying to formalize the concept further the class agreed that to fully describe the rational number system another postulate must be added) to permit division (except by zero). Further, since they were going to study only the rationals, it would be correct to write 7 in place of 7/1, always understanding that 7 was merely a convenient symbol for 7/1.

Now it was clear that the negative numbers had to involve still another extension of the number concept, an extension that would permit universal subtraction. But whatever the nature of the extension, all the postulates—including the division postulate—had to be retained.

The customary 1 to 1 correspondence between numbers and points on a line was established. Addition and subtraction provided little difficulty once their meaning was made clear. The students then agreed that the positive rationals seemed to correspond to the unsigned rationals so far as addition and subtraction were concerned.

Having mastered addition and subtraction, the class returned to the study of the equation that had started the entire discussion of signed numbers. However, when asked to supply equations, they contributed not only the original type but also such examples as

$$x+7=-3, \quad x-7=-15, \quad x-7=-2.$$

In all cases the axiom of addition was used. Some students suggested that the axiom of subtraction could be used equally well. After seeing that the results were the same in a number of examples, the class voted a preference for the axiom of addition. There was no insistence upon a set form. Instead, the students were asked to show that they knew the nature of the process.

Returning to consider equations like $6x=18$, the class asked what solution could be found for $-6x=18$. Most students were puzzled. Some pointed out that the quotient of two signed numbers had no meaning for them. All agreed that the new number system would be of greater value if there were definite rules for multiplication and division, as well as for addition and subtraction.

The problem was to obtain a rule for multiplication which would not disturb the results of arithmetic studied previously. Since the class had seen that in addition and subtraction the positive rationals seemed to correspond to the unsigned rationals, they decided that the product of two positive numbers should be another positive number. That is, $(+6)\ (+7) = +42$ appeared to correspond to $6 \cdot 7 = 42$. The class noted that $+42$ was not to be identified with 42, but merely that a correspondence could be set up between the positive rationals and the unsigned rationals. In short, it was decided that $(+6)\ (+7)$ should equal $+42$.

The class then considered the fact that $(+6) \cdot 0 = 0$ could also be written in the form $(+6)\ [(+7)+(-7)]=0$, which could be expanded by using the fifth postulate. A typical pupil response was: "Since $(+6)$

(+7) equals +42, then (+6) (−7) must equal −42 in order that the answer be zero—*and the answer has to be zero.*" After studying several other such problems the class arrived at the conclusion that the product of a positive number and a negative number is always negative.

Someone asked if the product of a negative number and a positive number is also negative. Some said "yes" because of the first postulate on commutativity. Others were not satisfied and asked for an example which would show the necessity of this rule. A student suggested [(+6)+(−6)](+7) and was able to carry out the argument in complete detail. The problem of multiplying two negative numbers was handled in the same way, using (−6)[(+7)+(−7)]. That is, since the discussion had already yielded the fact that (−6)(+7) = −42, clearly (−6) (−7) must equal +42 to make (−6) [(+7)+(−7)] equal zero.

At this point many were loath to accept the results of their own reasoning. Rather than regress to purely verbal or literary interpretations of their results, the class reviewed the fundamental idea involved in extending their number-concept: no conflict with previous results must be involved. Retracing the steps in the development so far, they agreed that their results must be true; for any other conclusion would be nonsense.

The law of signs for division of signed numbers followed immediately from the work in multiplication, e.g. (−8)+(−2) must equal +4 because (−2) (+4) equals −8.

The application of these ideas to the solution of equations was rather simple. Having mastered $\pm 6x = \pm 18$, the class immediately attempted equations of the type $4x - 9 = -3 - 2x$. Again there was no insistence upon a routine way of obtaining the solution.

The chief points to be noted in this summary of the course are:

1. All processes are based upon generalizations of arithmetic.
2. In every situation—except, perhaps, in the introduction of the four axioms—the students are asked to think for themselves, using arithmetic concepts as a basis.
3. The process of solving equations is not formalized to the extent usually advocated, though the axiomatic method is thoroughly presented and used.
4. Checking of solutions was offered by various students as an interesting discovery. It was exhibited and its importance discussed, but no attempt was made to force the children to adopt it.

This fourteen week course was given to all three beginning algebra classes in a small junior high school in Baltimore, Maryland. The average size of these classes was 28 children. The age range was 13 to 15 years. The I.Q. range for the three classes was 90–120, 80–110, and 80–100. The median I.Q. for each of these classes was 105, 95, and 90 respectively. Neither the background nor the future plans of these classes differed greatly from the norm. The majority expected to attend senior high school, but few expected to continue their formal education beyond senior high school.

The most notable feature of this course was the interest displayed by the children in extending the number concept and in applying their own results to equations suggested by themselves. In general the more imaginative students responded with greater readiness to the challenges offered by the course. At its best moments the course became a voyage of discovery. A number of students confessed they had never known that they could think so much or so well.

"Number Stories of Long Ago" by David Eugene Smith is available again for the first time in many years. Send your order with remittance (75¢ postpaid) to National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

The Unbroken Mystery of the Broken Number

By Paul R. Neureiter
State Teachers College, Geneseo, New York

Whereas the measurement of understanding in arithmetic has received some attention in recent years,[1] we do not possess comprehensive data which would allow us to draw something like a profile map of understanding as actually achieved under the present regimen of arithmetic instruction. It can be safely asserted, however, that there is a sharp drop in understanding as pupils, gingerly guided by their teachers, descend from the rather brightly lighted plateau of the natural numbers into the misty lowlands of the artificial numbers, which for elementary school pupils are confined to fractions.

The work in Grades V to VIII is dominated by the concept of the fraction. And yet, how little functional knowledge of it is there among the population that has passed the Eighth Grade! Ample evidence is available on this point. Glaring deficiencies in the area of fractions and per cent were revealed by the thorough investigation of the arithmetic knowledge of high school pupils by Orleans and Saxe.[2] In "An Analysis of the Performance of College Freshmen on Arithmetic" H. C. Ayre[3] reports mastery coefficients of 52% for fractions, 57% for decimals, and 32% for percentage. Similar results were reported by W. S. Guiler.[4] In testing teachers college freshmen, whose background in high school mathematics varied from zero to eight semesters, the present writer found that the simple operations with common fractions were missed by about a third of the testees and questions on per cent by about two-thirds of them.

One can react to these disappointing results with the kind of irresponsible defamation of the public schools which is one of the favorite indoor sports of the American people. But the teacher of mathematics cannot afford such flippancy of judgment. He will find the reason for the obvious teaching failure, at least to a considerable extent, in faulty methods, in a presentation of the subject that does not do justice to the great conceptual difficulties involved in fractions and per cent.

That the human race as a whole has had a hard struggle with fractions is evidenced by the slow growth of the idea of general fractions and the deliberate avoidance of fractions through the creation of sub-units. "The origin of such compound numbers as 3 yd. 2 ft. 8 in. is to be sought in the effort of the world to avoid the use of fractions," D. E. Smith wrote in his *History of Mathematics*.[5] The modern use of the fraction implies a multiplicity of meanings and is further complicated by the co-existence of common and decimal fractions. Buckingham summarizes the multifold meaning of a fraction in nine distinct statements grouped under the two headings: "1. The fraction is a number," and "2. The fraction is a relation."[6] A complete understanding of the fine conceptual distinctions involved here can hardly be achieved on the elementary level even with the best possible teaching methods.

[1] B. A. Sueltz, "The Measurement of Understandings and Judgments in Elementary School Mathematics," The Mathematics Teacher, 1947, 40: 279–84. B. A. Sueltz, "Measuring the Newer Aspects of Functional Arithmetic," *Elementary School Journal*, 1947, 47: 323–30.

[2] Jacob S. Orleans and Emanuel Saxe, *An Analysis of the Arithmetic Knowledge of High School Pupils*, The City College Research Studies in Education No. 2, New York, 1943, pp. 74–81, 102–105.

[3] *The Western Illinois State Teachers College Quarterly*, 19: 2, September 1939.

[4] W. S. Guiler, "Difficulties Encountered in Fractions by College Freshmen," *Journal of Educational Research*, 1945, 39: 102–115. "Difficulties Encountered in Percentage by College Freshmen," *Journal of Educational Research*, 1946, 40: 81–95.

[5] D. E. Smith, *History of Mathematics*, Vol. II: (Boston: Ginn, 1925), p. 209.

[6] B. R. Buckingham, *Elementary Arithmetic, Its Meaning and Practice* (Boston: Ginn, 1947), p. 246.

But a survey of current practices suggests possibilities for improvement.

The first block to understanding is the unusual symbolism whereby two numbers are combined to represent one value. And the two numbers are in a tug of war with each other, one growing with the value of the fraction, the other growing in the opposite direction. This is a complicated relationship for which some degree of readiness should be developed in the child's mind. Then there is the subtle difference in the symbolic implications of numerator and denominator, the former having an enumerative function similar to the use of natural numbers in counting, and the latter defining the size of the fraction without counting any parts in the particular fraction at hand. Because of this essential difference in meaning it would be best if fractions were first presented with denominators written entirely in words; and even later when figures are used, they should be printed in a type that will visibly distinguish them from the numerators. Finally there is the principle of equivalent fractions whereby every fractional quantity can be symbolized in a theoretically unlimited variety of different ways. The one-to-one correspondence between symbol and value, which the child has unconsciously accepted in his experience with integers, breaks down completely.

So long as the fraction is a number, it is conceived as part of unity and, therefore, has an absolute value, quite in keeping with the absolute values obtained by the counting of concrete objects. But when the fraction is used to indicate a relation, all absolute value dissolves; the relativity of numbers comes to the fore. Children are entirely unprepared for this fact. It must be very surprising to them, for example, that one half of a quantity can be very much less than one hundredth of another quantity. Up to this moment fractions had arranged themselves in their minds according to size but now the absolute value of a fraction does not count at all in establishing a scale of values. One can find arithmetic textbooks on the market where the two uses of the fraction, the one for a number and the other for a relation, are presented practically on the same page. This practice is difficult to justify in view of the essential differences in meaning. When the traditional pie is dissected, the process is one of true partitioning. But when one third of a dozen eggs is found, no egg need be broken; the fraction is used in an extended sense which is psychologically quite different from the original conception of the "broken number." Infinitely greater care than is in evidence now should be used by teachers to prepare children for the difficult mental operation in which two numbers are brought into relation with one another and with a second pair of numbers. Every time a fraction is employed in this sense the child deals with a proportion, a subject intensively studied in more advanced courses.

The arithmetic teacher has a number of semi-concrete aids at his disposal for demonstrating the absolute values of fractions and relations between them.[7] Pupils can make their own aids by dividing circles, squares or rectangles into equal parts.[8] It should be pointed out in this connection that the circle is superior to all other figures for the demonstration of the fraction as part of the unit. Having the highest degree of symmetry, the circle is the natural symbol for unity. On subdivision the circle yields an entirely different figure, the sector, whereas rectangles create other rectangles. The representation of unity by a rectangle lacks the absolute quality which is inherent in a circle, but children think of numbers, first of all, in absolute terms.

No matter how skillful the teacher may be in the use of these aids, he cannot demonstrate with them the totality of meaning that is in a fraction. This totality com-

[7] H. W. Syer and D. A. Johnson, "Aids to Teaching," THE MATHEMATICS TEACHER, 1949, 42: 317 ff.; 1950, 43: 130 ff.; 1950, 43: 177 ff.

[8] J. V. Hall, "A Self-starter Approach to Fractions." THE MATHEMATICS TEACHER, 1950, 43: 331–33.

prises a number of partial meanings which fit together functionally into an organic whole. Under the current system of instruction the partial meanings are taught in disjunction from one another: first, the fraction as part of the unit; then the fraction as a ratio; next, the fraction in the form of a decimal; finally two or three cases of percentage. The key to competence in these separate areas is supposed to be found in repetitive drill. No wonder, the result is often rote performance of poorly integrated computational tasks. Insight and understanding come, however, from the perception of patterns that correlate and unify the various aspects of meaning.

We are here confronted with a fundamental difficulty of instruction in elementary mathematics. A number of abstract relationships that receive detailed study in high school and college reach down into the arithmetic classroom of the upper grades. An everyday problem such as "If a pound of meat costs 94 cents, how much should you pay for 14 ounces of it?" involves the idea of a function, namely, a direct variation, which for two pairs of terms can be solved by a proportion. The conceptual situation is far from simple, as can be judged from the difficulties experienced in this area by older students. It is further complicated by the interaction of two systems of measurement and partition, one decimal, the other non-decimal. No amount of "motivation," such as a project in managing a butcher shop (which in itself is a good thing), will ever impart an understanding of the process and result in functional competence unless the abstract relationship is squarely faced. As we struggle to reform the teaching of arithmetic in the direction of more insight, we should, first of all, endeavor to draw the line between the abstract concepts that are fit for presentation on the elementary and junior high school level and those that are not. If the underlying abstraction is so complex that it cannot be adapted to the apprehensive power of the children, the topic resting on the abstraction should be thrown out of the curriculum. Conversely, if a topic appears in the curriculum as a subject for mastery, we must expect the teacher to be capable of bringing the abstract relationship within the bright circle of the child's comprehension. Because this is rarely the case, fractions and per cents remain essentially obscure to a large part of the population.

In searching for a unifying pattern which would bring together the various meanings of fractions and per cents, and which would provide a visual demonstration of them, the author has developed a device which may be called a "fractograph." (See Figure 1) It consists of two fraction wheels turning upon each other in the manner suggested by the circular slide rule. The base, composed of white plastic, shows all the commonly used fractions with denominators ranging from 2 to 32 and, in proper relation to these, a scale of per cents. On this base turns another circular disk of transparent plastic, which indicates the same divisions of the circle as the base, but only by marks, without numbers. The base disk demonstrates the mutual relations of size among the fractions and their per cent equivalents. The principle of equivalent fractions becomes self-evident from an inspection of the scales, printed on the base disk in a two-color scheme that distinguishes numerators from denominators. To perform the four fundamental operations with common fractions, the top disk is rotated in relation to the base. The opposing patterns of direct and inverse operations are reflected in a clockwise sense of rotation for addition and multiplication and the reverse movement for subtraction and division. The manipulator converts the arithmetical process he wishes to perform into a geometric experience with correctly apportioned sectors and thereby confirms the result of the arithmetical operation. Calculation is not solely a manipulation of symbols, but a visual experience in relative sizes corresponding to the symbols.

As the scales sub-divide the circle which represents unity, they illustrate the primary character of the fraction as a number.

In order to show the more complex use of the fraction as a relation (see *supra*), auxiliary numbers are written either on the transparent top disk or on paper around the periphery of the per cent scale. With the aid of these the three types of problems exemplified by the questions (1) What is 3/16 of 32? (2) 12.5 is what part of 50? (3) 25 is $\frac{1}{4}$ of what number? are visualized. The same pattern underlies the three cases in percentage; and it can be seen that it is by no means a simple pattern. Any number can become the base for a percentage problem; it would then appear on the "fractograph" as the endpoint of a scale running from zero around the circle to the given number. The percentage problem reveals itself in this context as a problem in the correlation of two scales, the one a fixed per cent scale, the other a variable scale dependent on the base. In their progress around the unit circle the two scales are directly proportional to one another and can be used, together with other similar scales, to illustrate proportions and the direct variation.

(*Continued on page 18*)

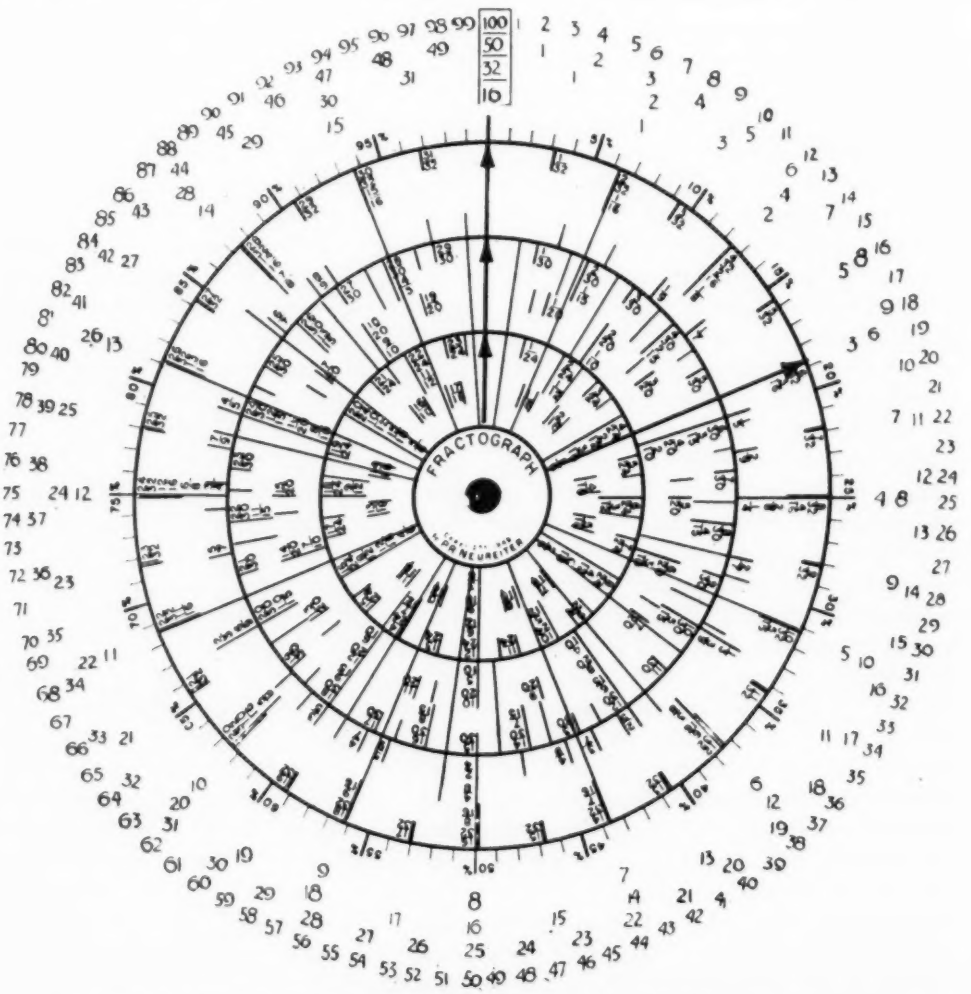


FIG. 1. 3/16 is a constant relationship between varying pairs of numbers.

Approximate Square Roots

By CARL N. SHUSTER
State Teachers College, Trenton, New Jersey

THE SLIDE rule is very satisfactory for finding square roots accurate to three significant digits. There are situations, however, in which a more accurate root is desired. The method given below will enable the operator to use his slide rule in combination with ordinary division or with a computing machine to find roots to 6, 8, 12 or more significant digits.

If we assume for a first approximation that the square root of a^2 is $(a+e)$, where e is the error in our estimate, then

$$a^2 \div (a+e) = a - e + a^{-1}e^2 - a^{-2}e^3 + a^{-3}e^4 + \cdots.$$

If the quotient is averaged with the divisor the result is:

$$a + \frac{e^2}{2a} - \frac{e^3}{2a^2} + \frac{e^4}{2a^3} - \frac{e^5}{2a^4} + \cdots \text{ (first ave.)}.$$

If a^2 is divided by the first average and a second average (quotient+divisor$\div$2) is obtained the result is:

$$a + \frac{e^4}{8a^3} - \frac{e^5}{4a^4} + \cdots \text{ (second ave.)}.$$

Since e is quite small in comparison with a and the signs of the terms alternate, only the terms $e^2/2a$ and $e^4/8a^3$ need be considered in estimating the approximate errors in the first and second averages.

The approximate error that results from using the first average for the root is $e^2/2a$, and the approximate error that results from using the second average for the root is $e^4/8a^3$. Since the error in using the slide rule is around 1 in 1,000, the error e obtained by using the slide rule will be in the neighborhood of $0.001a$. Substituting this value in $e^2/2a$ the error in the first approximation will be $0.0000005a$. This is so small that it will not affect the first five or six digits of the answer. If a five or six digit answer is sufficiently accurate, it will not be necessary to use the second average.

Substituting $e=0.001a$ in $e^4/8a^3$, the error in the second average is 0.000 000 000 000 1a. This is only one five-millionth of the error in the first average. The second average will produce an answer far more accurate than will be needed with the most accurate measurement that has ever been made. The very remarkable improvement of the second average over the first average is of considerable value when the first divisor is not as accurate as that obtained by using the slide rule.

The square root of 3 obtained by using the slide rule is 1.73.

$$3 \div 1.731 = 1.733102 \text{ (first quotient)}.$$

$$(1.731000 + 1.733102) \div 2 = 1.732051 \text{ (first ave.)}.$$

The square root of 3, accurate to 8 digits is 1.7320508. This rounds to the seven digit answer found by averaging the first divisor and the first quotient. The table below shows the remarkable accuracy obtained by using this method when only the first averages are used.

In the following table all the roots found as an average of the slide rule root and the first quotient are accurate to the first seven significant digits. Should greater accuracy be desired it is possible to secure this by dividing by the first average and averaging a second time. Doing this we obtain $\sqrt{5}=2.2360679772050$. The actual time required to find this 14 digit root using the slide rule and a calculating machine as a team was less than two minutes. This is certainly the most rapid and efficient method of obtaining square roots of this type at the disposal of the average person.

This same method may be used with cube roots or higher roots. The approxi-

Number	Slide Rule Root	First Quotient	First average (approximate root)	Root to 8 Significant Digits
5	2.235	2.237136	2.236068	2.2360680
7	2.644	2.647503	2.645751	2.6457513
10	3.16	3.164556	3.162278	3.1622777
26	5.10	5.098039	5.099019	5.0990195
30	5.48	5.474453	5.477226	5.4772256
56.42	7.51	7.512649	7.511325	7.5113246

mate cube root of 100 as found on the slide rule is 4.64. Now $100 \div (4.64)^2 = 4.64476813$ and

$$(4.64+4.64+4.64476813) \div 3 = 4.64158938$$

which is correct to at least seven significant digits. Slightly over one minute was required to obtain the above answer with the slide rule computing machine combination.

It should be pointed out that in finding the square root of approximate numbers by the above method, or any other method, the root should be rounded to the same number of digits as there were in the original number. It was assumed that the numbers used in this paper were exact numbers and hence the approximate roots could be carried out to any number of digits. Also, it was assumed that the divisors were as accurate as the quotient. The 4.64 used above would be 4.640000 if seven digits were to be retained in the answer. (The zeros were omitted to save time.)

If a computing machine is not available a combination of the slide rule and ordinary long division will be far more efficient than any pencil method. It might be noted also that this method is the most meaningful and the most easily rationalized. It is a very old method if not the oldest and uses no mathematics above long division. Once learned it is far less apt to be forgotten than any other.

The Unbroken Mystery

(*Continued from page 16*)

The comparison of fractions with constant numerator and variable denominators illustrates the inverse variation.

It is evident that these patterns are of a fairly high degree of complexity; and so long as figuring is confined to the manipulation of symbols on paper without any visual experience of quantitative relations, the possibilities for creating insight are meager, indeed. The remedy clearly lies in new and ingenious methods for giving the abstract concepts and relations a concrete, visual expression.

Mathematics Kits

A limited supply of sets of Straight-Line Unit No. 113 is still available. This kit discusses the interesting problem of drawing an original straight line without copying one. It contains material for assembling four workable models of linkages, with an explanatory leaflet.

Computation Unit No. 123 has been popular. This kit contains materials for assembling an addition-subtraction slide rule, a logarithmic slide rule, and a simplified set of Napier's Rods. It contains also a plastic slide rule and an explanatory leaflet.

All kits sell for 50¢ each or 3 for $1.00. Quantities may be assorted. Orders should be sent to the National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

Some Transformations from a Unit Circle with Line Values of the Sine Function

By FRANK G. DICKINSON

Director, Bureau of Medical Economic Research, American Medical Association, Chicago, Illinois

SEVERAL transformations from a unit circle were developed a few years ago by manipulating a physical model of a unit circle of 6 inch radius constructed to serve as a classroom aid for explaining the sine curve, the cycloid and the cardioid to students at the University of Illinois.[1] Eight views, V-1 to V-8, of the model are shown herein. The two semi-circumferences (figures 1 and 2 and view V-1) are hard rubber, $\frac{5}{8}'' \times \frac{5}{8}''$. The directed line segments MP are steel drill rods of $\frac{1}{8}$ inch diameter with their bent ends tightly fitted into holes drilled into the rubber circumference at arc intervals $\pi/24 \times 6''$ while the rubber was held in a straight line. (The rods in the upper half of the circle are in front and in the lower half, behind.) This arrangement of rods permits manipulation with

[1] The large number of Army and Navy trainees in 1943 forced the Department of Mathematics to borrow staff members from other departments to teach freshman and sophomore classes in mathematics. I was one of those loaned by another department. I started the tedious work on these models solely in the hope of finding a device for shortening the time required to explain the sine curve in trigonometry classes for the Army and Navy trainees. (See figures 1B and 2A.) When the model was finished, I was surprised to discover that it could be easily manipulated into a cycloid or a cardioid. Although I assume full responsibility for the ideas embodied in this article, I wish to acknowledge my great indebtedness to the following members of the Department of Mathematics at the University of Illinois in 1943: Josephine H. Chanler, Professor-Emeritus Arnold Emch, Henry J. Miles, Professor-Emeritus George R. Miller, J. William Peters and, most of all, Everett L. Welker. It seems unlikely that a mathematician would bother about making a physical model for plane trigonometry. I also constructed a rather clumsy model for graphing the tangent function and another for the secant function.

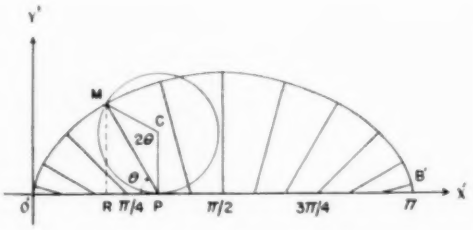


Fig. 1A Cycloid from unit circle

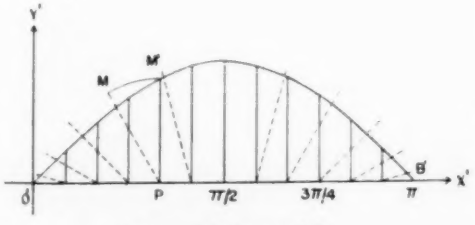


Fig. 1B Sine from cycloid

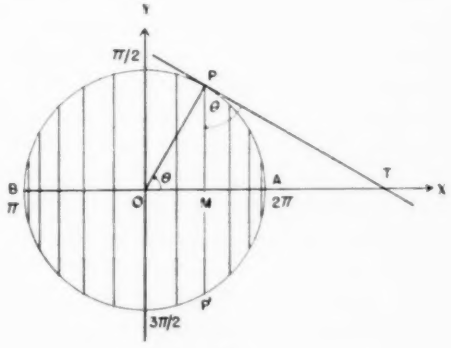


Fig. 1 Unit circle

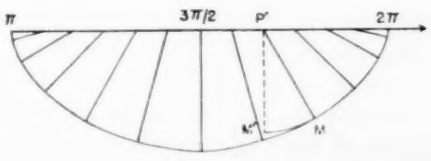
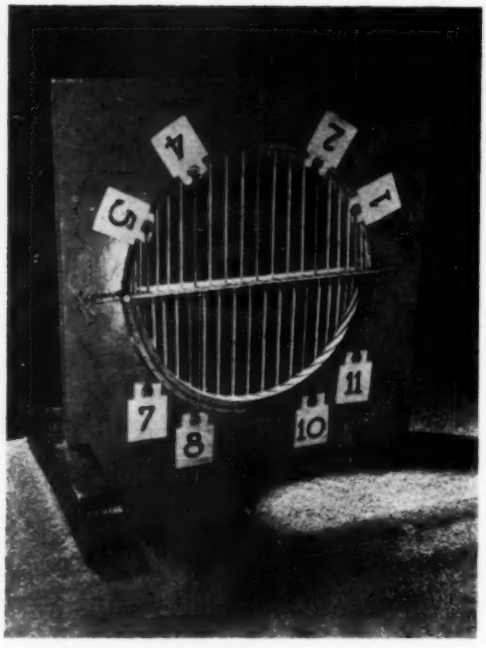


Fig. 1C Lower arch

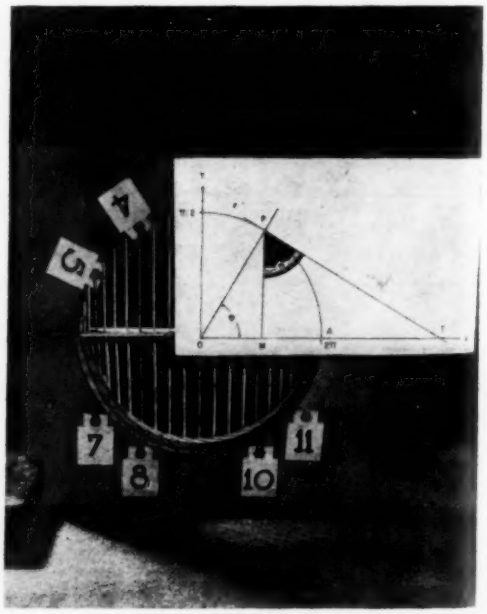


V-1. Unit circle.

very little change in the angle formed by the MP rods and the rubber (angle MPT in figure 1 and in view V-2). In order to hold the rubber in a circle it was squeezed into a hole ($r=6\ \frac{5}{16}''$) in a board, $18''\times 18''\times\frac{3}{8}''$. (See figure 1 and view V-1.) Another board with a hole of half this radius was used for the smaller cardioid. (See figure 3 and view V-8.) This physical model is commended to teachers of elementary courses in mathematics and physics. It can be made in a school workshop with wood, drill rods, a tractor fan belt of hard rubber and a narrow piece of steel for the center line. Conceivably it could be made of other materials. The eight views of the model, manipulated to match the five transformations presented in the figures, are intended to demonstrate the usefulness of the model to teachers. A tag marked "1" is attached to the $\pi/6$ rod, a tag marked "2" to the $\pi/3$ rod, one marked "4" to the $2\pi/3$ rod, etc. Although one cannot construct a straight line equal to the length of the semi-circumference of a circle, one can easily straighten out a semi-circumference made of rubber; this is the unique advantage of the model.

These five transformations from a (unit) circle are: 1. To a cycloid (figure 1A and views V-3 and V-4); 2. To a cycloid, then a sine curve (figure 1B and view V-5); 3. To a cardioid (figure 2 and view V-6); 4. To a cardioid, then to a sine curve (figure 2A and views V-7 and V-5); 5. To a cycloid, then to a cardioid (figures 1A and 3 and view V-8). Note that the arc intervals of the model are one half those of the figures $-\pi/24$ and $\pi/12$.

Each transformation starts with a unit circle (figure 1 and view V-1) with line values MP of sin θ with the angle θ in standard position. Preserving angle MPT ($=\theta$)—the black wedge in views V-2, V-3 and V-4—in the unit circle of figure 1, flatten out the upper semi-circumference by swinging counterclockwise until it lies on the $O'X'Y'$ axes of figure 1A and view V-4 so that A coincides with O', B coincides with B', and the ordinates MP become line segments in the first quadrant of the $O'X'Y'$ co-ordinate system. (View V-3 is an intermediate step to V-4; no counterpart is shown between figures 1 and 1A.) Angle θ, MPT in figure 1 and view V-2, is preserved as angle MPO' in figure 1A and view V-4; $O'P=$ arc θ of figure 1 (AP);



V-2. Unit circle. (Note black θ.)

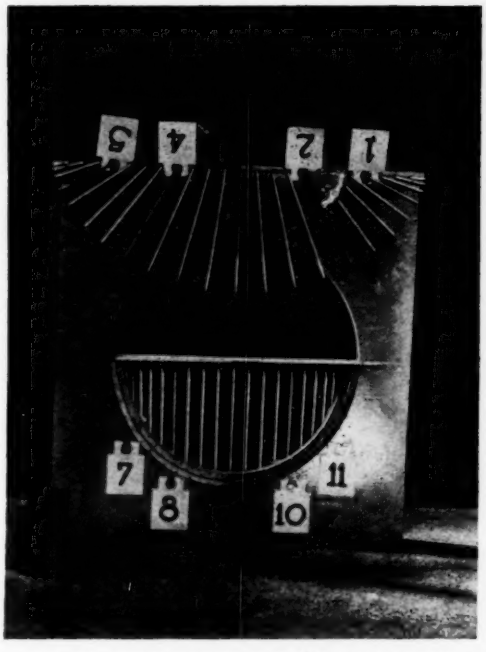
$MP = \sin\theta$. MR is perpendicular to $O'X'$ at R. M lies on the cycloid generated in figure 1A by rolling without slipping the circle with center at C of radius CP ($=\frac{1}{2}$ unit) along the $O'X'$ axis:

$$x' = O'P - RP = O'P - MP\cos\theta$$
$$= \theta - \sin\theta\cos\theta$$
$$= \tfrac{1}{2}(2\theta - \sin 2\theta)$$
$$y' = MR = MP\sin\theta = \sin^2\theta$$
$$= \tfrac{1}{2}(1 - \cos 2\theta).$$

Angle MPO' in figure 1A equals one half the angle of rotation MCP, 2θ, of the rolling circle.[2] MP is the *length of normal* to the cycloid at M. (In passing, it may be noted that the x'-projection of $MP = \frac{1}{2}\sin 2\theta$, that the y'-projection of $MP = \sin^2\theta$, and that the envelope of the family of normals of figure 1A is a congruent cycloid with a cusp at $[\pi/2, -1]$ and with maximum ordinates $[0, 0]$ and $[\pi, 0]$.)

The second transformation is shown in figure 1B, indicated in figure 1C and shown in view V-5. When the segments of normal MP are rotated into vertical positions $M'P$ about P, i.e., normal to the flattened out semi-circumference $O'B'$, M' lies on the sine curve:

$$y' = \sin x'.$$



V-3. Unit circle to cycloid. (Note black θ.)

[2] The transformation itself almost requires the notation 2θ, although it is awkward. Several students stated that the substitution for ϕ for 2θ obscured the relationships between the figures showing unit and half-unit circles, which are so prominent in most of these transformations.

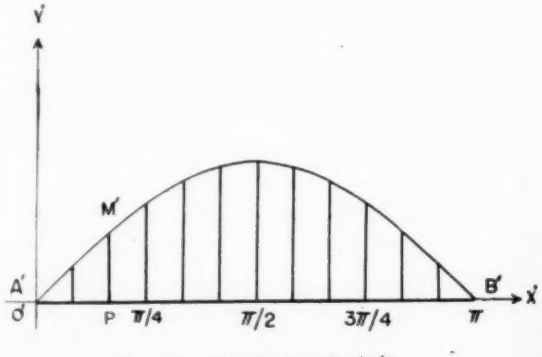


Fig. 2A. Sine from unit circle.

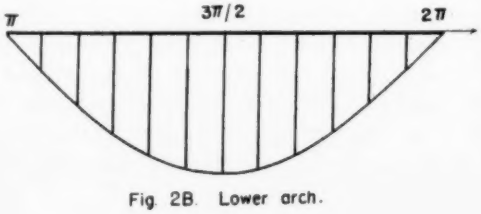


Fig. 2B. Lower arch.

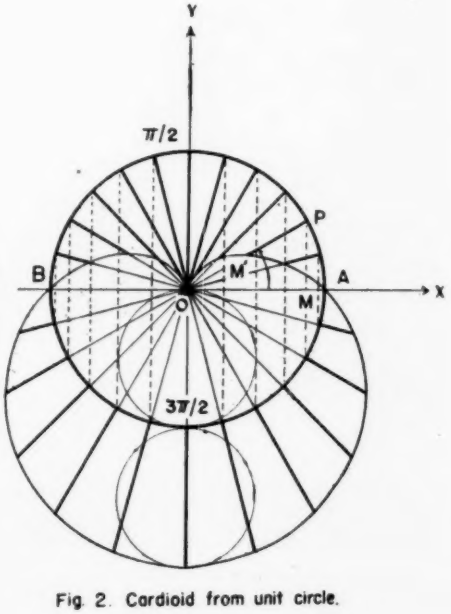
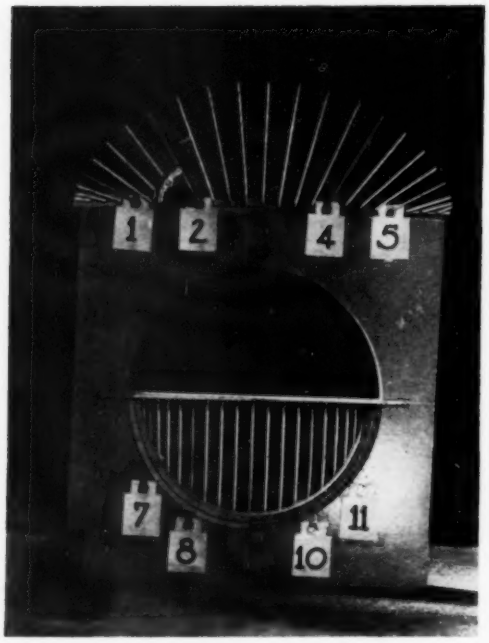
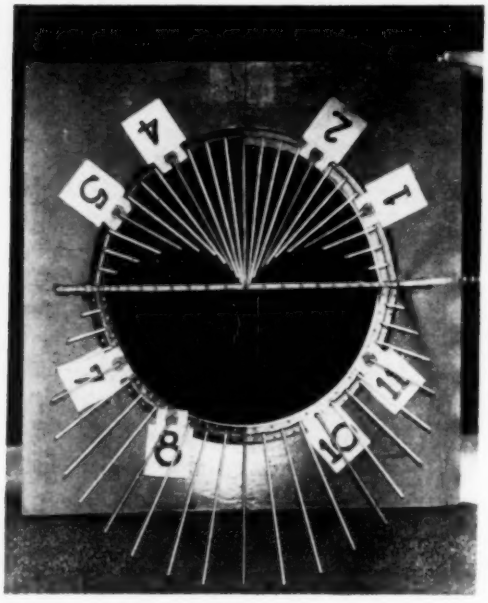


Fig. 2. Cardioid from unit circle.



V-4. Cycloid. (Note black θ.)

The third and fourth transformations start with the unit circle of figure 2. The directed line segment MP and the rods in view V-6 have already been swung through an arc about P until M' lies on the terminal side of θ (AOP) in standard position; M' lies outside the unit circle in the third and fourth quadrants. $M'P$ is then normal to the unit circle at P. This provides a point by point construction of the cardioid, $\rho = 1 - \sin\theta$, with pole at 0.



V-6. Cardioid from unit circle.

V-5. Sine from cycloid.

For the fourth transformation we flatten out the upper semi-circumference of figure 2 and of view V-6 and place it on the $O'X'Y'$ axes of figure 2A so that A coincides with O', B coincides with B' and the line segments are normal to the flattened out semi-circumference, $O'B'$. (View V-7 is an intermediate step and no counterpart is shown between figures 2 and 2A.) M' then lies on the sine curve:

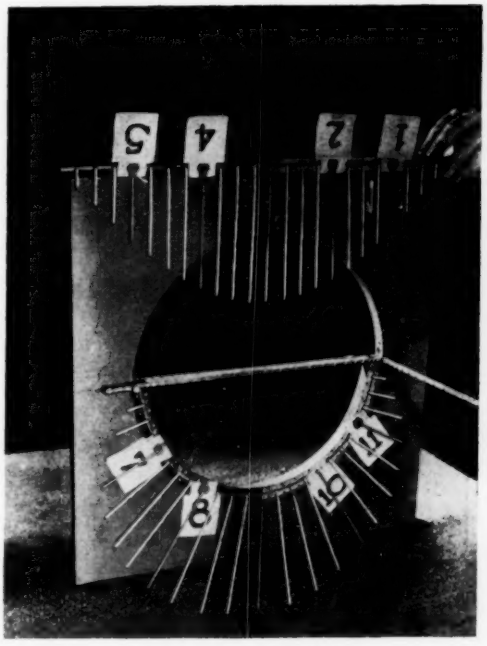
$$y' = \sin x'.$$

The fifth transformation starts with the circle with center C and radius CP of $\frac{1}{2}$ unit of figure 3. The cycloid developed in the first transformation of figure 1A and view V-4 is, after reversing $O'B'$ end to end bent around the circle C—hence doubling every angle θ from figure 1A and view V-4—so that O' and B' meet at O, the origin in figure 3, and the line $O'B'$ coincides with

the circumference of the circle C.[3] (Note view V-8.) Angle $OCP = 2\theta$ and arc $OP = \theta$. (For clarity of figures, the illustration angle θ chosen in figure 1 is $\pi/3$; in figures 2 and 3, $\pi/6$.) M now lies on a cardioid.

For the proof we roll without slipping the outside circle with center S ($r = \frac{1}{2}$ unit) of figure 3A around the circumference of the stationary circle C ($r = \frac{1}{2}$ unit) through the arc $OP(=\theta)$ with central angle 2θ. M'' by definition lies on the cardioid, $\rho = a(1 - \cos 2\theta)$ with origin at O; and $a = 1$. Draw line PT tangent to both circles at P.

[3] Line $O'B'$ (with lines MP) of figure 1A, after reversing $O'B'$ and to end, could be rocked without slipping counterclockwise around circle with center C of figure 3, unbent (straight) and tangent to produce the lines MP and the 11 points on the cardioid of figure 3. Since the cycloid is a symmetrical curve it is not strictly necessary in the figures to reverse $O'B'$ end to end but it is in manipulating the model; otherwise all the rods would lie behind the small wooden frame in view V-8.



V-7. Sine from cardioid.

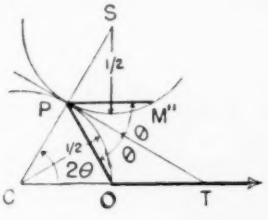
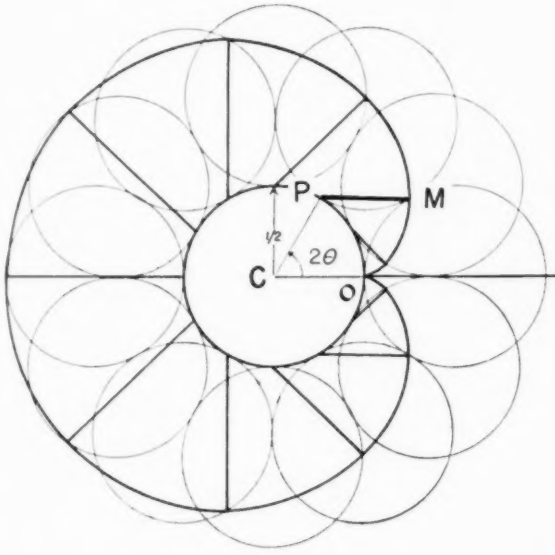


Fig. 3A. Cardioid by definition.



$\rho = 1(1 - \cos 2\theta)$

(origin at O)

Fig. 3. Cardioid from cycloid from unit circle.

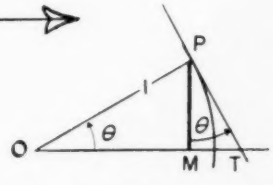


Fig. 3B. Unit circle.

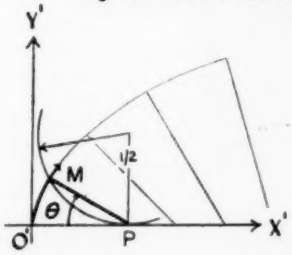
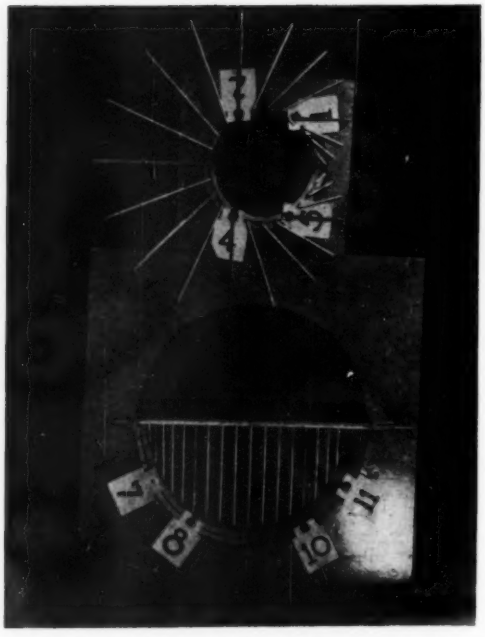


Fig. 3C. Cycloid from unit circle.



V-8. Cardioid from cycloid from unit circle.

Then angle $M''PT$ = angle $OPT = \theta$; chord $M''P$ = chord OP. Chord $M''P$ of this figure equals the half chord MP of the unit circle of figure 3*B* with θ in standard position, equals MP of figure 3*C* and hence equals MP in figure 3 ($MP = \sin \cdot \theta$). The first transformation to figure 1A is repeated in figure 3C for convenience with MP as the *length of normal* at M to the cycloid generated by the rolling circle of $\frac{1}{2}$ unit radius. Again, $M''P$ of figure 3A equals MP of figures 3C and 3; also, angle OPM'' of figure 3A and angles OPM and OCP of figure 3 are each equal to 2θ.

Five transformations from the unit circle have been described.[4] The first (to a cycloid) and the fourth (to a cardioid, then to a sine curve) involved flattening out the semi-circumference; the fifth transformation (to a cycloid, then to a cardioid) involved forming the original unit circle semi-circumference, after it had been flattened out (figure 1A), into a circle of $\frac{1}{2}$ unit radius coinciding with the circumference of circle C of figure 3, doubling meanwhile angle MPT ($=\theta$) of figure 1. Thus the cycloid, the sine curve and the cardioid can be produced by transformations from the unit circle (with half chords MP); and from each other. And it may be stated in rather loose terms that these three basic curves are different ways of "looking at" a unit circle.

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[4] Three additional transformations—numbers six, seven and eight—are interesting but not basic. These three are more interesting when achieved by manipulating the model. Six, a lower arch of the sine curve could have been produced by exploding outward and upward both ends, *A* and *B*, of the upper semi-circumference of figure 2 in order to flatten out this semi-circumference, as is shown in the upper part of view V-7 of the model. M' would then lie on a lower arch of a sine curve similar in most respects to figure 2B. Likewise rotating M' about P in the third and fourth quadrants until M' lies on the terminal side of θ, as in figure 2, except that M' lies inside instead of outside the unit circle, and then exploding outward and downward both ends of the lower semi-circumference—in order to flatten out this lower semi-circumference—would place M' on the upper arch of a sine curve similar in most respects to figure 2A. Thus the maximum (negative and positive) ordinates of the two arches would coincide at the origin, O. Could the inherent nature of wave motion conceivably involve this type of explosion—flattening out the upper and lower semi-circumferences as shown in figure 2 with M' within the unit circle—with or without rotation about the y and x axes? Such speculations added to my enjoyment of making and manipulating the model.

The seventh transformation involves a single manipulation of both semi-circumferences joined together at *B* on the cardioid but not at *A* of figure 2 or view V-6. When the circumference of the entire circle is swung clockwise about *A* until the entire circumference coincides with the positive x-axis, M' will lie first on the upper arch and then on the lower arch of the sine curve in conventional position.

The eighth transformation produces two arches of the cycloid. It involves a single manipulation of both semi-circumferences joined together at *B*, but not at *A*, of figure 1 (or of V-1) of the unit circle. When the circumference of the entire circle is swung clockwise about *A* until the entire circumference coincides with the positive x-axis, M of the upper half of the unit circle will lie on the first arch of a cycloid similar to figure 1A and M of the lower half of the unit circle will lie on the second arch of a cycloid.

Note: The author of this article has in preparation an Appendix which will provide views of the tangent and secant models mentioned briefly in footnote one, the views of the models for the three transformations described in footnote four and views of other models not otherwise mentioned. A teacher may obtain without cost one copy of the Appendix by writing to the author at 535 N. Dearborn Street, Chicago 10, Illinois.

How to Express the Remainder in Division

By Richard A. Winter
Central Junior High School, St. Cloud, Minnesota

Each new school year, hundreds of seventh grade mathematics teachers are faced with the same common problem—namely, how to express the remainder in division. This problem is especially acute in school systems where there are a number of different grade buildings. Since the mathematics program in the grades often varies from building to building, the pupils, necessarily, converge on the junior high teachers with as many different methods of expressing the remainder in division as there are grade schools in the system. The seventh grade teacher is then forced for the sake of uniformity to select, arbitrarily, one of the many forms and to discard the rest. This practice very often results in confusion and expressions of dissension among the pupils, and justifiably so, because the pupils have assumed that their teachers in the grades have taught them the correct form in writing the remainder. Then, suddenly, they are told that their method is no longer acceptable. Such a practice can only result in a lack of confidence on the part of the pupils toward their teachers.

Some thought needs to be given to determine which is the most effective and real way to teach the meaning of a remainder in division. The nature of the problem should determine the most effective method to use.

There are two very commonly accepted practices that have to do with the form in which the remainder is written. One is this form: 62 R 7; the other is $62\frac{7}{8}$. Neither should be taught to the exclusion of the other. To illustrate: If you wish to divide 14 apples among 4 people, then each person gets 3 apples with 2 apples left over. Expressed mathematically thus: $14 \div 4 = 3$ R 2. In this form, the disposition of the remaining two apples is uncertain. However, if these two remaining apples are to be shared equally, then it is expressed this way: $14 \div 4 = 3\frac{2}{4}$ or $3\frac{1}{2}$. In other words, the remainder now means that each person would get $3\frac{1}{2}$ apples, not 3 with the disposition of the remaining two uncertain (3 R 2).

To illustrate further: A farmer has 21 head of cattle which he wishes to divide equally among his four sons. Obviously, each would receive five head with one left over. It is fair to assume that, unless the sons butchered the remaining head, it would be impossible for each to get a fractional share of that one odd head. It is clear from the above illustration, that writing the remainder in fractional form ($5\frac{1}{4}$) is meaningless.

A third problem which illustrates a general usage involving the understanding of a remainder is as follows: A person in charge of a banquet has to arrange for the seating of 60 persons. Each table will accommodate 8 people. How many tables will be needed?

None of the above methods of expressing the remainder is satisfactory because they lack meaning in this particular situation. Certainly, the remainder expressed as a fraction is ridiculous, because the concept of having a fractional part of a table would lead to certain engineering difficulties. Social amenities prohibit the use of the other method of expressing the remander (R 4). It is inconceivable to assume that if there will not be enough people to fill an additional table that we completely ignore the necessity of providing adequate seating facilities for them. These people, even though they may be chronic late comers, are entitled to proper consideration. Therefore, this situation, as well as others similar to it, involves a separate and distinct interpretation of the remainder concept to give it significant meaning.

Since none of the aforementioned methods of expressing the remainder is appropriate in the third illustration, it is now necessary that a new concept be introduced to fit this particular case. Not new in the sense that it is original or different, but new in the sense that it gives a fuller meaning to an old problem. One must conclude that under certain conditions, a remainder has meaning only when it implies that an additional whole unit must be added. Many such situations occur in everyday living. In a sense, this method of handling the remainder is closely related to rounding off numbers with one slight deviation. In rounding off, we customarily drop the remainder and either increase the whole number or leave it as it is, depending on the size of the number dropped. Here we assume that no matter how much remains, it becomes necessary to add one whole unit to the integral part of the answer.

Another concept of a remainder which is closely related to the one just discussed can be illustrated by the following situation: If it takes two yards of denim to make a pair of jeans, how many pair can be made from seven yards of material? The remainder in this case would be handled in a manner opposite to the method employed in the problem concerned with the seating of people at a banquet. Since it is ridiculous to imagine anyone bothering to make a fractional part of a pair of jeans, the only possible way of interpreting the answer in this case would be to conclude that the remainder be ignored completely and that the answer in this situation, and others similar to it, must be a whole number only.

From these illustrations it is clear that the manner of expressing the remainder is important if the answer is to have meaning and reality.

One question which might be asked is, "What would one do with the remainder when the fundamentals of division are being taught and the pupils are engaged in the working of examples instead of problems?" The teacher should arbitrarily establish the procedure to be followed. This practice will not prove to be contradictory to an objection which was mentioned in the opening paragraph if the teacher uses a little skill and diplomacy in presenting the plan to be followed. First the pupils should be made aware of the variations in the way in which the remainder can be written. The teacher can easily explain that since no one form is always correct and that it would be confusing if each pupil were to decide for himself the form to use, the class might well agree to follow whatever plan were suggested by the teacher.

One might ask at this point, "How can such material be incorporated into the course of study?" At first glance it may seem impractical, but with a little effort it can be sandwiched in from time to time. It would be a waste of energy to even attempt to teach this material as one unit. The results of such material could only be worthwhile and meaningful if the topic were discussed and emphasized at strategic intervals throughout the year's work. Whenever the pupils are required to apply their knowledge of fundamentals in a problem solving situation, the teacher could very easily take each specific problem as it arose and, if it were of such nature that it required the use of a remainder, the teacher could discuss that particular problem as it applied to one of the aforementioned uses of a remainder. Using such a procedure would then, indeed, prove to be an effective approach to the administration of the program.

Five publications on financial security education are now available without charge from the **Committee on Family Financial Security Education,** 488 Madison Avenue, New York 22, N. Y. One is a resource unit designed for any grade of arithmetic from eight through twelve. Two bibliographies offered are "Some Supplementary Teaching Aids on Financial Security Education" and "A List of Motion Pictures and Filmstrips on Financial Security" which cover money management, banking, insurance, Social Security, investments, home ownership and consumer economics.

Mathematics: A Must for the Mechanic

By J. Douglas Wilson, *Supervisor*
Trade and Industrial Education, Vocational and Practical Arts Branch Curriculum Division, Los Angeles City Schools, Los Angeles, California

Introduction

The mathematics teacher often finds it necessary to go to the school shop and ask the shop teacher for help in the classroom. The mathematics teacher may need a table, a cabinet, or some device that will serve as a teaching aid.

Conversely, the shop teacher may ask the mathematics teacher to help the shop students with mathematics as applied to practical problems in the shop.

The purpose of this article, which is written by a shop man from a shop man's point of view, is to show the importance of coordinating the shop and the mathematics programs.

The Request for Mathematics

Business men, contractors, manufacturers, bankers, and merchants complain that their new employees do not know their "math." Apprenticeship committee members and employers of apprentices are constantly recommending that the apprentice be taught more mathematics. In California this is done in the school attendance period of four hours per week that is required by the state apprenticeship law.

An analysis of high school curricula shows that in many cases mathematics is not a required subject for graduation. Geometry and trigonometry are a *must* for the student who aspires to enter engineering or science courses in college, and shop mathematics is taught to some students who supposedly do not have the ability to take a more comprehensive course in mathematics. But many students do not take any mathematics course prior to graduation. Trade tests in the apprenticeship field also uncover the same deficiency in mathematics.

And so you have it! Industry complains that new employees are weak in mathematics; industry asks for more training in related mathematics, and yet many high school pupils are not required to take a basic course in the fundamentals of arithmetic.

Skilled Craftsmen Use Mathematics

Do educators recognize that the possession of mathematical skills is a *must* for the skilled craftsman? True, the operator of one simple machine doing nothing but repetitive work all day long in a factory probably never uses figures at all. Patterns are furnished for him to follow or, the job may be an assembly of parts, the proper holes and recesses already being located.

It must be asked, however, what about the foreman who directs the operative worker, or what about the layout man in the skilled trade? In both cases the possession of a wide variety of skills is a basic promotional requirement. One of these skills is a mastery of the mathematics of his trade.

Cold Storage Teaching

A basic teaching principle in vocational education is that of "teaching the unit of work or topic at the time the student needs it." In other words no "cold storage" teaching.

Actually, there is some "delayed usage" teaching, as it is not always possible to coordinate perfectly the related or technical instruction with the actual day-to-day work of an apprentice. The work will often change so rapidly that new knowledge and skills cannot keep pace with it. However, the interval between learning and using *should be kept to a minimum.*

CHART I

APPLICATION OF MATHEMATICS TO SKILLED TRADES*

FUNDAMENTAL TEACHING UNITS ← / SKILLED TRADES → (Apprenticeship program)	AUTO MECHANICS	BRICKLAYING	CABINETMAKING	CARPENTRY	ELECTRICIANS	MACHINIST & DIEMAKER	MEATCUTTING	PIPEFITTING	PLASTERING & LATHING	PAINTING & DEC.	PLUMBING	PRINTING	REFRIGERATION	SHEET METAL	CEMENT FINISHER
1. Whole numbers	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
2. Common fractions	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
3. Decimal fractions	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
4. Percent	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
5. Linear measure	X	X	X	X	X	X		X	X	X	X	X	X	X	X
6. Geometric constructions		X	X	X	X	X		X	X	X	X	X	X	X	X
7. Area	X	X	X	X	X	X		X	X	X	X	X	X	X	X
8. Volume	X	X	X	X				X	X	X	X		X	X	X
9. Weights and measures	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
10. Reading scale rule	X	X	X	X	X	X		X	X	X	X	X	X	X	X
Integrated: 11. Tables	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
12. Graphs	X				X	X		X		X	X	X	X		
13. Formulas	X	X	X	X	X	X		X	X	X	X	X	X	X	X
14. Equations (solving for unknown)					X	X		X			X		X		
15. Slide rule					X	X		X					X		
16. Trigonometry					X	X		X			X		X		

Acknowledgments →

CHART COPY EXPERTED BY:- Harry Bowe, Reaugh Fisher, William Hart, Carl Johnson, Ralph Rogers, Harry Simonds, John Strickley, William Walters, Apprenticeship Coordinators, William H Cole Apprenticeship Supervisor

MATHEMATICS TOPICS SUGGESTED BY: Dale Carpenter, Mathematics Supervisor, Curriculum Division, L.A. City Schools

CHART PREPARED BY J. Douglas Wilson, Supervisor, Trades & Industries, Vocational & Practical Arts Branch, Curriculum Division, 10-10-50, Los Angeles City Schools

*A representative list of skilled trades

In the case of mathematics this teaching principle, of necessity, cannot always be followed. However, the ability to use the fundamental arithmetical processes will speed up the preparatory training of a skilled mechanic. This mathematics training should be provided in advance of his entrance into trade training.

Application of Mathematics to Skilled Trades

A few months ago a group of apprentice coordinators in the Los Angeles City Schools were asked by the Supervisor of Mathematics to prepare a statement as to the need and extent of the use of mathematics in several specific trades. As a result of joint sessions of shop and mathematics teachers an analysis was made that provided graphic evidence of the extent of the use of applied mathematics in a number of representative skilled trades. Chart No. 1 shows the result of the conference.

Basic mathematics topics are listed in the column at the left of the chart. Each of the horizontal spaces represents a trade. The X's in each horizontal column indicate the mathematics topics that should be studied in preparation for the trade indicated in that column.

Extent of Mathematics Usage

A brief summary is outlined in Chart No. 2, showing the extent of usage in several mathematics topics, namely, whole numbers, common numbers, common fractions, decimal fractions, linear measure, and weights and measures. Each coordinator indicated that percentage was necessary in his trade, hence the per cent topic is not listed.

Responsibility of Mathematics and Shop Instructors

If it is accepted as a premise that more mathematics should be taught to high school students, whose responsibility is it

(*Continued on page 33*)

CHART 1

CHART 2. Extent of Mathematics Usage

Trades	Whole Numbers	Common Fractions	Decimal Fractions	Linear Measure	Weights and Measures
Automotive	Up to 4 digits	64ths	5 places	25 ft.	Troy-avoirdupois, linear, circular, liquid, time, temperature
Bricklaying	Up to 4 digits	16ths	2 places	1,000 ft.	Avoirdupois, circular, liquid
Cabinetmaking and Millwork	Up to 4 digits	32nds	2 places	1,000 ft.	Avoirdupois, circular, liquid
Carpentry	Up to 6 digits	32nds	2 places	1,000 ft.	Avoirdupois, liquid, angle measurements
Cement Finisher	Up to 4 digits	16ths	2 places	1 mile	Avoirdupois, liquid, angle measurements
Electrical	Up to 6 digits	64ths	4 places	300 miles	Avoirdupois, linear, circular
Machine Shop and Diemaking	Up to 4 digits	64ths	4 places	200 ft.	Metric, liquid, temperature, time, mechanical and electric power
Meatcutting	Up to 4 digits	16ths	3 places	None	Avoirdupois, liquid
Painting and Decorating	Up to 4 digits	32nds	2 places	1,000 ft.	Avoirdupois, linear, angle measurements
Plastering and Lathing	Up to 4 digits	32nds	2 places	1,000 ft.	Avoirdupois, liquid, angle measurements, mensuration
Plumbing	Up to 4 digits	16ths	4 places	10,000 ft.	Avoirdupois, time, troy, liquid
Printing	Up to 4 digits	72nds and fractional parts	4 places	25 ft.	Avoirdupois, liquid
Refrigeration	Up to 7 digits	64ths	8 places	10,000 ft.	Troy, liquid, avoirdupois, time
Sheet Metal	Up to 4 digits	32nds	4 places	1,000 ft.	Avoirdupois, linear, liquid, mensuration
Steam and Industrial Pipe Fitting	Up to 7 digits	64ths	8 places	10,000 ft.	Troy, avoirdupois, liquid, time, temperature

A Mathematical Dr. I. Quiz-em Program*

By The Mathematics Classes of Julia Diggins
Paul Junior High School, Washington, D. C.

Announcer: Mathematics Incorporated presents Dr. I. Quiz-em, the mental banker, master of wit and information. We hope you will enjoy his friendly smile and the cash, for your correct answer.

Dr. I. Quiz-em: Thank you, Mr. Announcer. We greet you from the stage of Paul Junior High School, the finest school, with the hardest working faculty, and the most scholarly students in the Nation's Capital. To allow the contestants to remain at their seats, we have placed Mr. Addition on the right aisle downstairs, Mr. Subtraction in the right center, Mr. Division in the left center, and Mr. Multiplication on the left aisle. When called on by the assistants, will the contestant please stand up and speak clearly. If at any time any member of the audience can not hear, will he please raise his hand.

(Before each question Dr. I. Quiz-em calls on an assistant in this manner):

Dr. I. Quiz-em: Mr. Addition in the right aisle.

Mr. Addition: I have a lady, Doctor . . . or (I have a gentleman), Doctor. (They do not call on the teachers.)

Dr. I. Quiz-em: For the correct answer, I will give that lady (or gentleman) _____silver dollars.

(Dr. I. Quiz-em often interrupts to say to the audience . . . "No prompting please." . . . or to a contestant, "That prompter gave you the wrong answer." . . . or, "Don't be nervous, the microphone won't hurt you." . . . or, "Five more seconds, please." . . . "Come on, take a guess." . . . or, "Shall I repeat the question?"

If the answer is correct, he asks the name of the contestant, congratulates him, and repeats the number of silver dollars promised for the correct answer. If the answer is incorrect, he says, "Oh, I am very sorry but a practice pad to that lady (or gentleman) for a very good try."

The Prizes are the promised number of silver dollars, to be collected at the end of the performance. The silver dollars are, of course, imitation. The practice pads are actual, but quite small, and given to the contestant by the assistant at the time of the answer.

Questions, like those shown below, are used.

1. What are the four fundamental processes in mathematics?
2. Name three plane geometric figures . . . (remember they are flat, and have only two dimensions).
3. Name three solid geometric figures . . . (remember they have a third dimension).
4. What does a commission of 10% on $85.40 equal?
5. Why did Mr. Smith build his pig pen 4′ high, 6′ wide, and 12′ long?

Dr. I. Quiz-em: Oh, I am sorry, I think you will find it was to put his pigs in. Pause. Now we will have a word from our announcer.

Announcer: (Read like a commercial.) I want to tell you about that simplest, and most invaluable contribution of mathematics. Even the smallest child appreciates it when he sees the candles on his birthday cake . . . that wonderful boon to man . . . is . . . ORDERLY ARRANGEMENT BY NUMBER . . . Yes, your age, 12, 13, 14, 20, 30, etc. Your city streets, first, second, third, etc., and your

* A quiz program given at Paul Junior High School, which followed as closely as possible the pattern of the Dr. I. Q. program on the radio. The goal to be achieved by the program was to stimulate greater interest in the subject of mathematics, greater interest in the history of mathematics, and the great personalities among mathematicians, and greater interest in quick mental response to mathematical problems.

house numbers. What would the post office do without them? The numbers of the floors of a building, and the room numbers . . . if it were not for these, think of the people who would be growing long white beards while they looked for offices in the Empire State Building! Consider your section numbers, 7A1, 8A2, 9B4, etc., the pages of a book, the sizes of your clothes, and I'll bet each one of you is thinking of a dozen other uses of this delightful, delicious, little aid to finding things. . . .

Dr. I. Quiz-em: Thank you, Mr. Announcer. Now we will go on to the next question.

1. Who invented our modern system of numbers, and what are these numbers called? (Hindu-Arabic numerals)
2. Which is cheaper, 2 pounds of onions for 19¢, or 3 pounds for 24¢?

(A particular member of the audience has been selected for this answer.)

Contestant: ER . . . er . . . er . . .

Dr. I. Quiz-em: Now don't be nervous . . . give us an answer . . . the first thing on your mind.

Contestant: Er . . . I've forgotten the question. I'm scared.

Dr. I. Quiz-em: I will be glad to repeat it. Which is cheaper, 2 pounds of onions for 19¢, or 3 pounds for 24¢.

Contestant: Well, you see, I don't like onions.

Dr. I. Quiz-em: My dear man, that is beside the point. Will you please choose the cheaper of the two prices?

Contestant: The first.

Dr. I. Quiz-em: I am very sorry, but I think you will find it is the second, 3 pounds for 24¢. But a practice pad to that gentleman.

Contestant: Now, I never will like onions.

3. How many inches in 2 feet?
4. How many square inches in 2 square feet?
5. What is the area of a square with a side 5 feet long?
6. Name one important Greek Mathematician in history.

Dr. I. Quiz-em: Now we come to the biographical sketch of a famous personality, sent in by Mr. Decimal Point. We have already sent to Mr. Decimal Point a prize, a book of great social significance entitled "Nothing Rolls Like a Ball" or "Get In Your Right Place and Stay There." Others who sent in questions on the same interesting personality will receive a practice tablet. We selected the one by Mr. Decimal Point because we felt it was presented in the clearest manner to enable you to recognize this week's character. You will get $75 if you recognize him on the first question and $10 less on each succeeding question and if you fail to identify the person as the questions progress, the money will be sent to Mr. Decimal Point. Are you ready for the first question? Don't hesitate, guess if you are not sure, but try to give some answer to each question.

1. He was born in Florence, Italy in 1564.
2. His love of mathematics led to his fame as a scientist.
3. His contributions to science were so fundamental that the centuries have done little but amplify and adapt them to myriad uses in our industiral life.
4. While watching the swinging chandelier in the Cathedral of Pisa he was inspired to formulate the laws of the pendulum.
5. He constructed a telescope that enabled him to make important astronomical discoveries.
6. He considered his most important work the proof of the theory of Copernicus, that the earth revolves around the sun.
7. He performed an experiment that led to his law of falling bodies by dropping weights of unequal size from the leaning tower of Pisa.
8. His name was_____.
(answer: Galileo)

If you wish to send in an entry for a biographical sketch send in your entry with four 100% mathematics papers to

Dr. I. Quiz-em, Room 301, Paul Junior High School, Washington, D. C.

Dr. I. Quiz-em: Now, we interrupt for a few words from our announcer.

Announcer: (Read like a commercial.) What else does mathematics do for us every day? Well, it measures things. Our day begins and is planned by the clock, the measurer of time. How many times during the day do you say, "What time is it?", or look longingly at the clock to see when the lunch bell will ring? Your dress, your clothes are all accurately measured to size. You eat food that is accurately measured by a recipe. And "Oh Mother, please don't get the wrong measure of salt in that beautiful lemon meringe pie you are making!" You go to the store and buy a pound of butter, a quart of milk, a bushel of apples, a yard of cloth, etc. You run a $\frac{1}{4}$ mile in track and what is more you owe all our modern conveniences to the art of exact measurement.

Dr. I. Quiz-em: Thank you, Mr. Announcer. Now for more questions.

1. Are there more 7's in 280 than there are 9's in 450?
2. How much dirt is there in a hole 1′ wide, 1′ long and 1′ deep? (Oh, I am very sorry, I think you will find there is no dirt in the hole.)
3. Which of the following three words does *not* refer to mathematics: octahedron, octahedrite, or octant?
4. If a pole stands 5′ out of water, is $\frac{1}{4}$ under water and $\frac{1}{3}$ more sunk in the mud, how long is the pole?

Dr. I. Quiz-em: Now we come to the set of right or wrong statements. A check for $200 has been sent to Mr. Quadrilateral for these six statements. Those in the radio audience will not be eligible for the prize, but they can get out their pencils and try their skill. The answer must be yes or no and only one trial is possible.

1. A hexagon has eight sides.
2. A trapezoid has one pair of parallel sides.
3. The circumference of a circle is pi times the diameter.
4. If a horizontal line crosses a vertical, the lines are perpendicular.
5. An acute angle has more than 90 degrees.
6. The sum of the angles of a triangle is 360 degrees.

A contestant wishing to enter this contest must send in five 100% mathematics papers with every set of questions. Now for more questions!

1. Name three early civilizations that contributed to the knowledge of mathematics?
2. What is the volume of a cube whose edge is 2″?
3. Add 6 and 2, divide by 4, subtract 1, multiply by 5.
4. Say in another way: 12:45, 12:30, and 12:15.
5. What do you call lines that never meet?

Now we come to that monument to memory, the thought twister, which I will say once, and one time only, so listen carefully, for I will not repeat. Will the audience please be quiet and give the contestant a real chance!

"Arithmetic is easy," said Teasy to Sneasy, "Arithmetic is easy, Teasy," to him said Sneasy.

Again we will hear from our announcer, but stay tuned for the famous quotation question which we will give in a few minutes.

Announcer: I've told you some of the everyday things mathematics does for *you*. Now, what do *you* do everyday for mathematics? If you appreciate pages following one another in correct order, and the accuracy of correct time, correct sizes in your clothing, correct measurement in buying materials, in building houses, correct change in the store, then be sure to try to achieve accuracy in your everyday mathematics work.

Dr. I. Quiz-em: Thank you, Mr. Announcer. Here is another question.

1. What kind of an angle is made by the hands of the clock when it is 12:20? 12:05? and 6:00?

2. Give the formula for the area of a rectangle.
3. I have $1.20. If I divide it equally among 10 boys, how much will each one get?
4. Multiply 3 by 6, subtract 2, divide by 8 and add $\frac{1}{4}$.

Now we come to that gem of wisdom, the quotation . . . Since last week's contestant failed to recognize the quotation, we will add what he missed to your winnings to-day. If you give me the correct answer I will give you $200 . . .Who said, "Eureka! I've found it!"?

(The assistants were warned not to request any of the teachers to participate in the program. But one of them could not resist picking from the standing audience the assistant principal, who, of course, won the silver dollars.)

Mathematics: A Must for the Mechanic

(Continued from page 29)

to see that this is done? Here is really the $64.00 question. Undoubtedly different answers will be given by different persons. However, the responsibility may be divided as follows:

1. Let the shop instructor canvass his field and prepare the facts regarding the need for and use of mathematics in his shop subject.
2. The mathematics instructor must next be sold on the validity of the information presented to him by the shop instructor. Then he should present the facts to the proper administrative authority. The school administrator should then be willing to adapt the curriculum to meet the needs of students in the community and thus provide the solution to the problem.

Summer Fellowships for Secondary Teachers of Mathematics

The General Electric Company has established an annual mathematics fellowship program under which secondary school mathematics teachers may take summer courses at **Rensselaer Polytechnic Institute** for a six-week period beginning July 7 and continuing through August 16, 1952. Dr. Livingston W. Houston, president of Rensselaer, recently announced that 50 fellowships will be awarded to teachers from 12 eastern states and the District of Columbia. The program is designed to familiarize secondary school mathematics teachers with the application of mathematics to the various fields of human endeavor. It will be conducted by the Rensselaer faculty in co-operation with General Electric scientists and engineers. The program will provide classroom and laboratory instruction for the teachers, in addition to various trips through G-E plants and laboratories in the area and will be under the direction of Dr. Edwin Brown Allen, professor of mathematics and head of the Department of Mathematics at Rensselaer.

The grant of each fellowship by the General Electric Company includes tuition, fees, maintenance at Rensselaer during the six-week period, and traveling expenses. To be eligible for one of the 50 fellowships, a secondary school mathematics teacher must have experience in the field, hold at least a bachelor's degree representing substantial undergraduate courses in mathematics, and teach within designated states. These states are Maine, New Hampshire, Vermont, Massachusetts, Rhode Island, Connecticut, New York, New Jersey, Pennsylvania, Delaware, Maryland, Virginia and the District of Columbia. The awards will be made by a special committee set up by Rensselaer Polytechnic Institute and the announcement of the 1952 awards will be made in the spring.

This mathematics fellowship program is the third fellowship program that the General Electric Company has established. Each year 100 secondary school science teachers are awarded fellowships for a six-week summer period at Union College, Schenectady, N. Y., and Case Institute of Technology, Cleveland, Ohio.

WHAT IS GOING ON IN YOUR SCHOOL?

Edited by JOHN R. MAYOR *and* JOHN A. BROWN
The University of Wisconsin, Madison, Wisconsin

LOS ANGELES CITY COLLEGE MATHEMATICS PRIZE COMPETITION

IN THE spring of 1951, Los Angeles City College held its first annual William B. Orange Mathematics Prize Competition for high school students of Los Angeles City High Schools. The competition was inaugurated as a memorial to the late William B. Orange, formerly Chairman of the Department of Mathematics of Los Angeles City College. It consisted of a two hour written examination held on May 25, 1951. Schools were permitted to send up to two teams of three students each, with a resulting total of 161 students from 33 high schools. The team prize, won by Eagle Rock High School, with Dorsey High School a close runner-up, was a 40 inch bronze trophy. This trophy is to be presented annually to the winning school. Eight individual prizes consisting of cash, slide rules, books and professional magazine subscriptions were awarded to students from the following high schools: Eagle Rock, Dorsey (2), Washington, Belmont, Fairfax, University, and Hamilton. Ten additional students were awarded mathematics handbooks and six more students were listed for honorable mention.

Questions used for the competition were:

Part I: (50 per cent)

If any problem is impossible, write "no solution" for answer. Each problem counts 2½ per cent; partial credit will be given if problem has more than one answer.

1. Find two integral factors (other than unity and the number itself) of 1,000,001.
2. What is the missing digit in the following products

$$(15348456)(76144846) = 11687058-8457776?$$

3. Simplify:

$$6-\cfrac{5}{2-\cfrac{4}{1-\cfrac{1}{4}}}$$

4. Simplify:

$$|\sqrt{(-i)^3}|,$$

where $i=\sqrt{-1}$.

5. Express as a fraction: $0.32222222\cdots$
6. Solve for x:

$$\begin{vmatrix} x & 17 & -4 \\ -6 & 2 & 12 \\ 4 & -11 & -8 \end{vmatrix} = 0.$$

7. Solve for real values of x: $10^x+(3)10^x=4$.
8. If $\log_{10}2=0.30103$ and $\log_{10}3=0.47712$, find $\log_{10}2.5=?$
9. $$2^{\left[\frac{1}{\log_3 2}\right]}=?$$
10. Arcsin (sin 2) = ?, if the angle is in the first quadrant.
11. Evaluate:

$$\frac{\cos 150^\circ \tan 300^\circ}{\cot 225^\circ+\sin(-30^\circ)}$$

12. Write the eighth term of the binomial expansion of:

$$(x/2+2/x)^{14}$$

13. The area of a rectangle is 40. Find the diagonal in terms of the perimeter alone.
14. If the roots of $ax^2+bx+c=0$ are x_1 and x_2, write the equation whose roots are $-1/x_1$ and $-1/x_2$.
15. The sides of a right triangle are in geometric progression. What is the ratio of the progression?
16. A seven is written at the end of a number, thereby increasing the original number by 700. What was the original number?
17. What is the maximum number of points of intersection of 3 straight lines and 3 unequal circles? (include intersections of lines with lines, circles with circles, and lines with circles).

18. Suppose a new decimal clock were introduced. Each day of 24 hours would be divided into 100 hours, each hour into 100 minutes, each minute into 100 seconds, with the day starting at midnight. What would be the new time if an old style clock read 4:30 P.M.?
19. If we multiply together all the numbers from one up to and including 100, how many zeros will be at the end of the product?
20. A circle is inscribed in an equilateral triangle, and a square is inscribed in the circle. What is the ratio of the area of the triangle to the area of the square?

Part II: (50 per cent), 1 hour

1. (15 per cent)
   Prove: The distances of any point on a diagonal of a parallelogram to the adjacent sides are inversely proportional to the lengths of those sides.
2. (15 per cent)
   (a) A statue ten feet high stands on a pedestal eight feet high. How far from the pedestal is a point on the ground where the statue and the pedestal subtend equal angles?
   (b) If the heights of the statue and pedestal are a and b respectively, for what values of the ratio a/b, if any, is the problem unsolvable?
3. (20 per cent)

$$1^3 = 0 + 3 - 3 + 1$$
$$2^3 = 1 + 12 - 6 + 1$$
$$3^3 = 8 + 27 - 9 + 1$$
$$4^3 = 27 + 48 - 12 + 1$$
$$5^3 = 64 + 75 - 15 + 1$$
$$6^3 = -125 + 108 - 18 + 1$$
$$\vdots$$
$$n^3 =$$

   (a) Write the expression for n^3 in the above series of identities and prove it to be true.
   (b) Using your result show that the sum of the squares of the first n positive integers is

$$\frac{n(n+1)(2n+1)}{6}$$

   (c) Describe how this process could be generalized to find the sum of kth powers of the first n positive integers.

Report of the Competition and the questions used were submitted by BEN K. GOLD, Competition Committee Chairman, Mathematics Department, Los Angeles City College.

MATHEMATICS IN BASEBALL

Twenty-six fifth and sixth grade children, attending The University of Wisconsin 1951 Summer Laboratory School, found a common interest in baseball. They had, however, given little thought to the use of measurement as it related to this phase of their daily play. Thus, "Mathematics in Baseball" became the center of interest during their six weeks together. It should be added that the area of interest had been previously defined in the over-all planning of the school program, that this program is not remedial, and that activities are so selected to provide enriched and new experiences for the children as they work together in solving their problems.

As in any group, there were varying degrees of interest in the aspects of the problem. Too, these children, coming from twelve schools and three states, had different previous experiences. Thus, following discussions of this problem, individuals and small groups chose to study topics of special interest to them, bringing their findings to the class in chosen ways as radio or television skits, panel discussions, quiz programs, talks illustrated by slides, film strips, and models.

What were some of the interests? (1) Making scale models of ball fields (softball, baseball—junior and major leagues, football) gave the class the basis for a comparative study of the fields. As one child said, "Now I see why I get tired when I make a homerun." (2) Laying out a softball field for the class to play on provided a problem through which the right angle was introduced to the group. (3) Learning of the origin of standard units of measure interested others. From their reports, the class became interested in their own spans and paces, comparing them with other members of the class and using them for approximating chosen distances which they later measured. (4) Interested in the measurement of time, one group recorded information concerning the history of the calendar on lantern slides for presentation to the class; others made models of old timepieces. Following

the explanation of standard time zones, members of the class assumed positions across the States, "setting their clocks" accordingly. Naturally, this created much interest in the problem of travel around the world. (5) Discussion of the "magic umpire" found one group building a model to show the scientific principles involved in the "magic eye," (6) Meaning of batting averages carried out to our own ball games where we learned to keep a simplified score card of games played. Also, the group attended together one league game being played in the city for the purpose of collecting similar statistical information. Both situations provided much material for interesting mathematical problems.

It is true this study did not have the same appeal for all children. Such remarks were overheard—"But this is not arithmetic. When do we get our books out?" . . . "We have so much arithmetic during the year—we don't want anymore." . . . "I like this. We do so much arithmetic and don't have to use a book." If these children had been working together previous to this time, they would probably have chosen to work more closely together as they did in another area later. No doubt, in most cases, this would be more desirable. Certainly any of these interests offers much opportunity to give meaning *for* some of those mathematical concepts that one already has meaning *of* or which was developed in the instances of using units of measure, batting averages and graphical representations.

E. Glenadine Gibb
Iowa State Teachers College
Cedar Falls, Iowa

Daily Assignments

In *The Wisconsin Teacher of Mathematics* a number of statements from teachers have been published, by invitation, on what they do about daily assignments. The following two statements were taken from the February, 1951, issue:

"Do you ever have difficulty in getting students to prepare their assignments? If so, this method may help you as it did me.

Subject ———— Date ————

I have not prepared my assignment for today because

———————————
Signature

"When a student came to class without having prepared his assignment, he was given a form like the above on which to state his reason. After about four slips were collected they were sent to his parents with a special report. This was very effective in securing parent cooperation.

"I found this device particularly helpful in teaching students who were repeating subjects. I felt that it was a very important factor in preventing failures."

Lydia R. Goerz
Mary D. Bradford High School
Kenosha, Wisconsin

"I have found that difficult home work assignments are discouraging to pupils and lead to copying, or having someone else do the work for them. Since a great deal of drill work is needed in mathematics, I usually assign some examples for which previous written work has shown a need, or drill work on a new idea, such as an added step in equations. I usually try to have a few at the end of the lesson which will be a challenge to the bright student, and which the slower student can omit if too difficult for him.

"In class we go over all of the examples which were missed by several people; sometimes I have these put on the board by the pupils who missed them so I can help them where they need help; sometimes I send the good student to the board so that the slower student can see how it should be put down and worked; and sometimes I put the examples on the board myself, explaining the steps, or having pupils explain them as I put them down.

"In our school, we require homework of our seventh and eighth graders on only one evening so that, although the pupils have homework every night, one night it will be English, another night mathematics, still others social studies and science. This is not a burden for them, but does get them into the habit of spending an half hour or so on school work in the evening. In ninth grade, most pupils have five or more study periods a week, so assignments are made in all academic work.

"I seldom correct the home work but go over it very thoroughly so the pupils can correct their mistakes; but I do go over their papers sufficiently afterward so that I can see that the process has been understood, and the lesson finished."

Ruth E. Bartholomew
Central High School
Madison, Wisconsin

Garfield Mathematics Club

For a number of years Garfield students who had a keen appreciation of mathematics in its many phases had been investigating interesting sidelights of the field of mathematics and related subjects, but they found that time for discussion of such topics during the class period was too limited to be satisfactory. In 1948 it was decided that a mathematics club would furnish the opportunity for the desired study and discussion; so the Garfield Mathematics Club was organized. At first the plan was that only juniors and seniors taking algebra III, solid geometry, college algebra, or trigonometry were to be eligible for membership, but later provision was made to include, by special permission of the sponsor, those sophomores who showed promise and interest. During the first year only boys joined the club since the Girl Reserves met at the same time and usually all the girls of the school belonged to the latter organization. But since that time, several girls have chosen the mathematics club in preference to the Girl Reserves.

Our club meets once a week during the activities period, which is one half hour in length; so the programs have to be well planned in order that everything can be taken care of during that short time. Since the club is not large, comprising from twenty to twenty-five members, active participation during the semester is possible for all members. Of course, we have more than that number in our junior and senior mathematics classes, but Garfield has a wide variety of clubs for advanced students, some of which meet on the same day as the Mathematics Club.

The programs of our club are varied and are planned and carried out by the students assisted by suggestions from the sponsor. At the end of the first year a vote was taken to ascertain which type of program was preferred, the recreational kind or student reports which usually require quite a bit of research. The vote was almost unanimous in favor of the latter. The members usually choose their own topics, thereby following their particular interests but in some instances, the program chairman assigns the subject for the report. A short question and discussion period follows the presentation of each topic.

Reports given have included such topics as: The Global Concept, The Theory of Relativity, The Fourth Dimension, The Mathematics of Probability and Gambling, The World Calendar Reform Plan, The History of Logarithms, Napiers Rods, Topology, Magic Squares, The Binary and Five Number Systems, Linkages, and Non-Euclidean Geometry. The last named was discussed for the club by a student teacher.

This year, in a series of programs, the club has been studying the history of surveying instruments. A few of the boys are planning to make some of the simpler instruments which can be added to those already being used by the geometry classes in field work. Several of the earlier meetings each year are devoted to the study of the slide rule. There seems to be no end to the possibilities for subjects for discussion. In preparing their reports the students use the Public Library, the Garfield Library, our own mathematics library and other sources.

On the recreational side, puzzles, brain teasers, paradoxes, and fallacies are used for program material. We already have quite a collection of mathematical puzzles. some of them having been made by members of the club. These formed part of the Garfield exhibit at the Sci.-Math. Assembly at Purdue University last spring. The Mathematics Club planned and organized this trip and the exhibits.

Among the outside speakers for club programs have been Mr. Carnahan of Purdue University, Dr. Wilkinson, President of Rose Polytechnic Institute, and Mr. Wheeler, Vice President in charge of production at the Commercial Solvents.

Since no club is complete without social activities, two parties are held each year,

the Christmas Party during the first semester and a picnic for members and guests at the end of the year. As a money making project, the club sponsors a mixer in the school hall after one of the basketball games. Proceeds from this mixer and the dues, which are fifty cents a year, provide the only source of revenue for the club. Plans are being made this year to use part of the club funds to purchase books and materials for the mathematics library.

As sponsor of the Mathematics Club, I feel it is serving its purpose well, and I have found it a pleasure to work with the group because of the interest and enthusiasm displayed. In order to give a picture of the value of the club from the students' point of view, several members have expressed their opinions as follows:

"Being in the Mathematics Club at Garfield has given me a feeling of accomplishment. Some of the clubs one might join are purely social, but the Mathematics Club has combined fun and education. The Mathematics Club stimulates our minds by introducing new types of fields in which a knowledge of mathematics plays an important part; it provides entertainment through the brain teasers and puzzles while developing the mental processes; it shows how mathematics can be applied in later life and the relationship between it and other subjects—especially the other sciences. Because the Mathematics Club does give you a feeling of having accomplished something while having a good time, not only the boys, but we girls, too, look forward to our regular meetings each week."—Regena Houpt

"The Mathematics Club has given me a better understanding of mathematics in general. I feel I have received more information on things of interest than in the regular classroom. The club also goes into the less serious side of mathematics. I have enjoyed working on the fallacies, puzzles, and the games based on mathematical formulas. Being a member of the club also helps me to meet people with the same interests and gives me the opportunity of discussing these interests with them. The club should mean a good time for any student who likes mathematics."—Charles Hayward

"I sincerely appreciate the opportunity of being a member of the Garfield High School Mathematics Club. Since first joining the organization over two years ago, I have gained much knowledge not only of mathematics but in related fields also. This experience in the Mathematics Club, I am sure, will be of great assistance to me when I continue my education in college. One of the many values which the Mathematics Club is to the student is the opportunity which the student is given to express himself when he gives his individual report to the club. Also, the student is given a much greater opportunity for further study in specialized fields of mathematics. The theory of relativity, the fourth dimension, the laws of probability, the slide rule, and topology—to mention only a few—were studied by the students with a great benefit to all. Furthermore, the speakers who came from the various colleges were of great help to the students in the way in which we were enlightened on the subjects for advanced study of mathematics in college. All of these values combined in a spirit of cooperation are sure to provide fun for everyone concerned. But, perhaps, the greatest dividend which we gained from our work in this organization has not yet been mentioned. From our association with mathematics teachers in this organization, we were able to overcome that age old superstition that teachers of mathematics were supernatural, and we found them to be real live human beings."—Paul Charles Elliott

(Paul Elliott, Garfield senior, was one of the eleven state winners in the Westinghouse Science Talent Search.)

Reported in *The Indiana Mathematics Teacher*, official newsletter of the Indiana Council of Teachers of Mathematics, March 1951, by Inez Kelly, Garfield High School, Terre Haute, Indiana.

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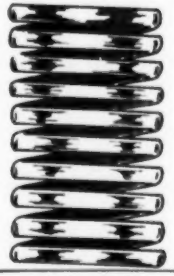
$$L = \frac{N^2 R^2}{9R + 10\frac{N}{n}}$$



L = Inductance in Microhenries
N = Total number of turns in coil
R = Radius of coil (in inches)
n = Number of turns per inch

The solution of this Problem is Simplified by Applying the above Formula It Becomes: —

$$R^2N^2 - \frac{10L}{n}N - 9RL = 0$$

Solve for "N"

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POSTER #15

COMPLEX QUANTITIES IN ELECTRICITY

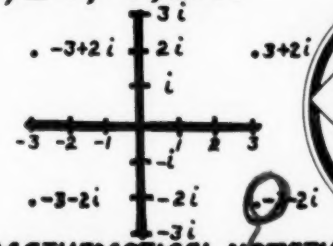
($\sqrt{-1}=i$ IN MATHEMATICS; —— $\sqrt{-1}=j$ IN ELECTRICITY)

In the complex plane real numbers are represented on the horizontal axis. Imaginary numbers are represented on the vertical axis.

In electricity j is used in place of i to avoid confusion with the symbol i for current.

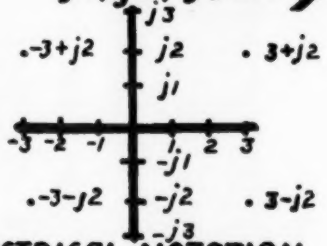
NOTATIONS

MATHEMATICS
i, 2 i, -i, etc.



MATHEMATICAL NOTATION

ELECTRICITY
j 1, j 2, -j 1, etc.



ELECTRICAL NOTATION

Points not ON either axis represent sums of a real and an imaginary number (complex numbers).

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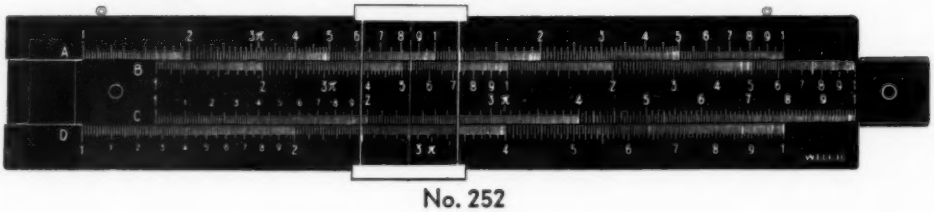

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MATHEMATICAL MISCELLANEA

Edited by PHILLIP S. JONES
University of Michigan, Ann Arbor, Mich.

39. A Geometric Proof for a Number of Trigonometric Theorems

A number of pupils in my trigonometry class asked whether it were not possible to prove the formulas for sin $(x+y)$ and cos $(x+y)$ by means of a simpler diagram than that given in the text book. In answer to that question, I developed the following proofs. Using one simple diagram, I found it possible to prove the formulas for the following:

$$\sin(x\pm y),\quad \cos(x\pm y),\quad \tan(x\pm y),$$
$$\cot(x\pm y),\quad \sin 2x,\quad \cos 2x,$$
$$\tan 2x,\quad \cot 2x,\quad \sin \tfrac{1}{2}x,\quad \cos \tfrac{1}{2}x,$$
$$\tan \tfrac{1}{2}x,\quad \cot \tfrac{1}{2}x.$$

I use the following diagram(Fig. 1). For simplicity let $CD=1$ unit.

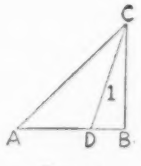


FIG. 1

Triangle ABC is a right triangle, right angle at B. CD is any line.

If CD is drawn so that $\angle DAC=\angle ACD=x$, then we can prove the formulas for sin $2x$, cos $2x$, tan $2x$, cot $2x$.

Thus, to prove sin $2x=2\sin x\cos x$, we let $AD=CD=1$, $\angle CDB=2x$: Let $CB=m$, and $DB=n$. Then sin $2x=m$. But

$$AC=\sqrt{(1+n)^2+m^2}=\sqrt{1+2n+n^2+m^2}$$

and since $n^2+m^2=1$, $AC=\sqrt{2+2n}$, then

$$\sin x=\frac{m}{\sqrt{2+2n}},$$

$$\cos x=\frac{1+n}{\sqrt{2+2n}},$$

which leads to

$$2\sin x\cos x=\frac{2m(1+n)}{2(1+n)}=m.$$

Finally, then, $\sin 2x=2\sin x\cos x$.

If we let $\angle DAC=\angle ACD=(1/2)x$, we can prove the formulas for sin $(1/2)x$, cos $(1/2)x$, tan $(1/2)x$, cot $(1/2)x$.

Thus to prove the formula for cos $(1/2)x$, we have

$$\cos(\tfrac{1}{2})x=\pm\frac{1+n}{\sqrt{2+2n}},$$

or

$$\cos^2\tfrac{1}{2}x=\frac{(1+n)^2}{2(1+n)}=\frac{1+n}{2}.$$

But $\cos x=n$, and therefore,

$$\cos\tfrac{1}{2}x=\pm\sqrt{\frac{1+\cos x}{2}}.$$

If CD is any line, we can prove the formulas for sin $(x\pm y)$, cos $(x\pm y)$, tan $(x\pm y)$, cot $(x\pm y)$. Thus to prove the formula for tan $(x+y)$, we let $\angle DAC=x$, $\angle ACD=y$, $BD=n$, $BC=m$, $AD=r$, $AC=s$.

$$\tan x=\frac{m}{r+n}.$$

By the Law of Sines, we have

$$\sin y=r\sin x=\frac{rm}{s}.$$

Also by the Law of Cosines, we have

$$\cos y = \frac{s^2+1-r^2}{2s} = \frac{rn+1}{s},$$

since $s^2 = r^2+2rn+n^2+m^2 = r^2+2rn+1$. Then

$$\tan y = \frac{\sin y}{\cos y} = \frac{rm}{rn+1}.$$

But

$$\frac{\tan x+\tan y}{1-\tan x \tan y} = \frac{\frac{m}{r+n}+\frac{rm}{rn+1}}{1-\frac{m}{r+n}\cdot\frac{rm}{rn+1}} = \frac{m}{n}.$$

Since $\angle BDC = x+y$, we have

$$\tan (x+y) = \frac{m}{n} = \frac{\tan x+\tan y}{1-\tan x \tan y}.$$

We can prove the formula for $\tan (x-y)$, by letting $\angle BDC = x$, and $\angle ACD = y$.

The above derivations can be assigned as supplementary problems. The pupils observe that, although the diagram is much simpler than the traditional one, the algebra is more involved.[1]

BENJAMIN BOLD
Boy's High School
Brooklyn, N. Y.

40. A Farewell (?) to Redians, Nedians, Cevians.[2]

MERTON T. GOODRICH'S notes on this topic to which we referred previously [cf. THE MATHEMATICS TEACHER, vol. 44, (May 1951), p. 311] were too detailed to print here. However, a summary of his results, and an indication of his methods may be a good conclusion to this series of notes. The following are excerpts from his paper titled *Redians*.

A *redian* is one of a set of lines drawn from the vertices of a polygon to the first opposite side, counting counterclockwise, provided the segments of each side cut off by the redians have the same ratio.

Our study begins with triangles but the author has proved:

(1) In any polygon the redian figures (polygons formed by the redians and sides) adjacent to the perimeter are of only two kinds, triangles and quadrilaterals.

(2) In any polygon the central redian figure is the same kind as the original figure.

(3) In any triangle, rectangle, square, or regular polygon the redian triangles (except the central redian triangle of a triangle) have the same area.

(4) The redian quadrilaterals of any triangle, parallelogram, rectangle, square, or regular polygon (except the central redian parallelogram of a parallelogram) are equal in area.

The author has derived formulas for the areas and their ratios of all the redian figures in terms of the coordinates of their vertices for triangles, parallelograms, rectangles, squares and regular polygons.

The triangle $P_1P_2P_3$ (Figure 1) has redians P_1P_4, P_2P_5, P_3P_6, intersecting each other in the points P_7, P_8, P_9. This triangle may be placed in a rectangular coordinate system so that the vertex P_1 is at the origin and the side P_1P_2 coincides with the X axis. Then, if the length of the segment starting at P_1 is a, and the ratio of the whole line to the segment is m, the coordinates of P_2 are am and 0, while the coordinates of the other vertex may be any numbers b and c. In any position, the altitude of the triangle on P_1P_2 is c. The coordinates of the points and the equations of the lines will be dependent upon the position of the triangle relative to the origin and axes, but the areas of the redian figures are not changed by any motion of the triangle. Therefore, the relations between the areas, established by the use of these coordinates, will be true for any triangle in any position.

Our first objective is to find formulas for the areas of the redian figures of a triangle, and the ratios between these

[1] *Department Editor's note:* A similar diagram and derivation is to be found in Roscoe Woods, "The Trigonometric Functions of Half or Double an Angle." *American Mathematical Monthly*, XLIII (1936), 174.

[2] Earlier discussions of these topics are to be found in "Mathematical Miscellanea," 18, 26, 32, 34, 37.

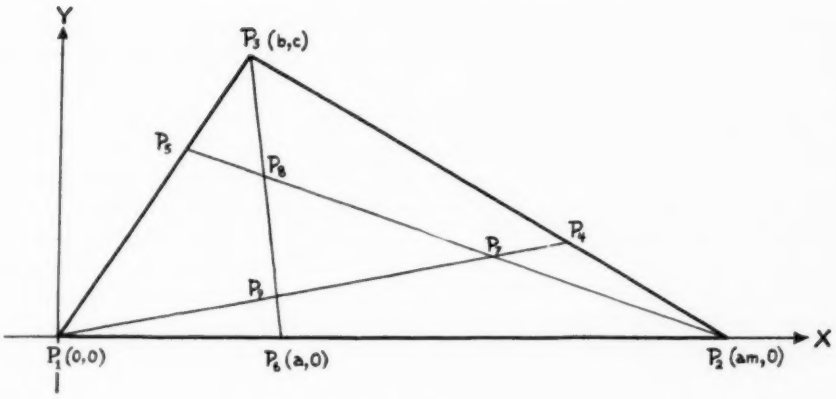


FIG. 1

areas. We follow the usual procedures of analytic geometry.

The x coordinate of the point of division of a line is given by

$$x=\frac{x_1+rx_2}{1+r}$$

where in this case

$$r=\frac{a}{am-a}=\frac{1}{m-1}$$

which finally leads to

$$x=\frac{x_1(m-1)+x_2}{m}\,.$$

Correspondingly

$$y=\frac{y_1(m-1)+y_2}{m}\,.$$

From this the coordinates of P_4, P_5, P_6, the redian points, are found to be

$$\left(\frac{am(m-1)+b}{m},\ \frac{c}{m}\right);$$

$$\left(\frac{b(m-1)}{m},\ \frac{c(m-1)}{m}\right);$$

and (a, o) respectively.

It is now possible to write the equations of the redians and find their intersections P_7, P_8, P_9, the vertices of the central redian figure. From the coordinates of these points the lengths of the redians and their segments are easily determined.

To compute the area of the redian triangle $P_7P_8P_9$ the array shown below is set up.

The common factor $1/(m^2-m+1)$ may be removed from each column, and the remaining array may be expanded by taking the algebraic sum of the pairs of the pairs of products as noted elsewhere.[3]

[3] Although this array has the form of a matrix it is actually a computation scheme for

$$\triangle P_7P_8P_9=\frac{1}{2}\begin{vmatrix} \dfrac{am^2-am+b}{m^2-m+1} & \dfrac{c}{m^2-m+1} \\ \dfrac{(m-1)(am^2-am+b)}{m^2-m+1} & \dfrac{c(m-1)}{m^2-m+1} \\ \dfrac{b(m-1)^2+am}{(m^2-m+1)} & \dfrac{c(m-1)^2}{m^2-m+1} \\ \dfrac{am^2-am+b}{m^2-m+1} & \dfrac{c}{m^2-m+1} \end{vmatrix}.$$

Finally then

$$A=\frac{1}{2}\cdot\frac{amc(m-2)^2}{m^2-m+1}$$

which, since $\frac{1}{2}\ amc=\triangle P_1P_2P_3$ reduces to

$$\triangle P_7P_8P_9=\frac{(m-2)^2}{m^2-m+1}\cdot\triangle P_1P_2P_3.$$

Note that both of these formulas are independent of b, thus indicating that the size of the redian figures depends solely on m and the area and not the shape of the original triangle.

What conditions result from different values of m? If $m=0$ there is no triangle $P_1P_2P_3$. If $0<m<1$ the redians divide the sides of the triangle externally. If $m=1$ the redians coincide with the sides, if $m=2$ the redians become medians and the central triangle becomes O which checks with the fact that medians intersect in a point. In this latter case the areas of the vertex triangles are each $\frac{1}{6}$ of the original area.

MERTON T. GOODRICH
Keene Teachers College
Keene, New Hampshire

41. *Another Outlook on Geometric Progressions*[4]

In all algebra text books when geometric progressions with infinitely many terms are treated, it is demonstrated that if the common ratio r is a number between -1 and $+1$, the sum S_n of the progression approaches the limit $a/1-r$ where a is the first term of the progression.

"How can the 'sum' of an endless succession of terms have a limited, finite value?" is often a baffling question in the minds of mathematics students at this stage. This confusion in the minds of students will be reduced if, before treating geometric progressions with infinitely many terms and using the related technical terminology, e.g., "as n, the number of terms, increases without bound, their sum S_n approaches the limit," one analyzes the simple problems of two mobile objects (two trains, or motorcycles, or airplanes, or the two hands of a clock) moving in the same direction with uniform but different speeds. These problems, which can be solved either by arithmetic or algebra and consequently are taught in junior high schools, will naturally be welcomed by senior high school students, who will hear them with understanding.

Let us consider one of these problems. Two trains, T_1 and T_2 leave at the same time from two stations, A_1 and A_2 one mile apart and run with uniform but different speeds in the same direction. If the velocity of T_1 is double that of T_2, what will be the distance covered by T_1 when it reaches the train T_2?

In order to solve this problem by algebra, let us call x the distance A_2R (Figure 2). Then $1+x$ and x will be respectively the distances covered by T_1 and T_2 at the time of their meeting. Considering that the distance covered by T_1 is the double of that covered by T_2, we can write $1+x=2x$ which gives $x=1$ mile. Consequently the answer is 2 miles, since A_1A_2 is also given equal to one mile.

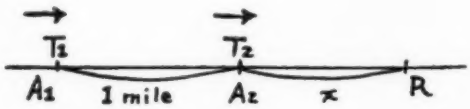


FIG. 2

It is worthwhile to notice that 2 miles as given by algebra is not an approximate value but an exact one.

Now let us observe the following facts with respect to the distances covered by the two trains (Figure 3). While T_1 covers the distance $A_1A_2=1$ mile, and arrives at

---

evaluating plane rectilinear areas. It is based upon determinantal notions. Some indication of these notions is to be found in "Mathematical Miscellanea," THE MATHEMATICS TEACHER, XLIV (Oct. 1951), 400.

[4] See "Mathematical Miscellanea 27," THE MATHEMATICS TEACHER, XLIV (May 1951), 311–12 for related comments.

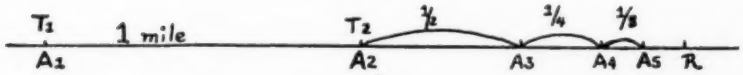


FIG. 3

A_2, T_2 covers the distance $A_2A_3=1/2$ mile and arrives at A_3, since its velocity is the half of T_1. But while T_1 next covers the distance $A_2A_3=1/2$ mile and arrives at A_3, T_2 in its turn is traveling the distance $A_3A_4=1/4$ mile and reaching A_4. Similarly while T_1 leaving A_3 covers this 1/4 mile and arrives at A_4, T_2 once more will escape and reach the point A_5 at 1/8 mile from T_1. It seems that this "catch me if you can" game of the two trains will be endless, but we know that it is not so and have already calculated by algebra, A_1R, the distance to the meeting point of the two trains as being exactly 2 miles.

Introducing now the idea of limit, we can write:

$$\lim_{n\to\infty}\left(1+1/2+1/4+1/8+1/16+1/32+\cdots+\frac{1}{2^{n-1}}\right)=2 \text{ exactly} \quad (1)$$

where the terms of this succession are the values of successive distances covered by T_1.

If we take the same problem in its general aspect, we will say that the separating distance of the two trains at the beginning of our observation of their movement is a and that the velocity of T_2 is equal to k times of the velocity of T_1. But since x and $a+x$ are respectively the distances covered by T_2 and T_1 when they meet each other, therefore x will be k times as great as $a+x$. We can write then

$$k(a+x)=x$$

or,

$$x=\frac{ka}{1-k}\cdot$$

Therefore, A_1R, the distance covered by the train T_1 will be:

$$A_1R=a+x=a+\frac{ka}{1-k}=\frac{a}{1-k}\cdot \quad (2)$$

This formula for the general case shows that we always have an answer to our problem, that is the two trains meet each other if $1-k\neq 0$ or $k\neq 1$. If $k>1$, the meeting distance is negative, which means that they will not meet in the future but may be interpreted as implying that they had already met each other (point R' on Figure 4) prior to the time when we started our observation, and finally if $k<1$, the meeting distance is positive and they will meet in the future.

These findings are almost self-evident, since if the speed of T_2 is greater than that of $T_1(k>1)$ their separating distance will be greater and greater and they will never reach each other; if T_1 and T_2 have the same speed ($k=1$) they will always maintain the same separating distance and again will never reach each other, and finally, if T_2 has a smaller speed than T_1 ($k<1$) then assuredly they will always meet each other.

In this general form the succession of the spaces covered by T_1 will be:

$$a,\ ak,\ ak^2,\ ak^3,\ \cdots,\ ak^{n-1}\cdots \quad (3)$$

and if $k<1$, the only case in which the two moving trains reach each other, A_1R the distance covered by T_1 at the very moment of its meeting with T_2 will be:

$$\operatorname{Lim}_{n\to\infty}(a+ak+ak^2+ak^3+\cdots+ak^{n-1})=\frac{a}{1-k} \quad (4)$$

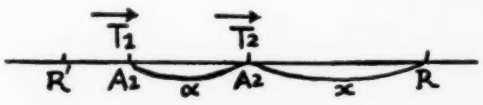


FIG. 4

We see clearly that the values of the above succession of distances covered by T_1 form a geometric progression. Now, and only now, the teacher without fear of being misunderstood by his students can treat geometric progressions with infinitely many terms and establish all the facts and their related formulas that we already know very well as the outcome of rigorous demonstrations.

It is worth mentioning that while the formulas for geometric progressions give values as a limit, the very same formulas established by algebra indicate the exact values.

But it can be protested that we have made our algebraic conclusions implicitly supposing that k was a positive number, while in the study of geometric progressions with infinitely many terms it is stated that the limit of the sum of its terms is $a/1-k$ not only when $0<k<+1$, but also when $-1<k<0$, therefore the algebraic conclusions are incomplete.

Let us consider this case of $-1<k<0$. If we suppose $k=-r$, r will naturally be a positive number, and the usual geometrical progression with infinitely many terms,

$$a+ak+ak^2+ak^3+ak^4+\cdots=S_n$$

can be written:

$$a-ar+ar^2-ar^3+ar^4-ar^5+ar^6-ar^7+\cdots-ar^{2p-1}+ar^{2p}+\cdots=S_n$$

Then, we can also write:[5]

$$(a-ar)+(ar^2-ar^3)+(ar^4-ar^5)\cdots+(ar^{2p}-ar^{2p+1})+\cdots=S_n$$

or

$$a(1-r)+ar^2(1-r)+ar^4(1-r)+\cdots+ar^{2p}(1-r)+\cdots=S_n$$

then,

$$(1-r)(a+ar^2+ar^4+ar^6+\cdots+ar^{2p}+\cdots)=S_n$$

But the second factor of this expression forms a geometrical progression with infinitely many positive terms whose sum is given by algebra as being exactly

$$\frac{a}{1-r^2}$$

since $0<r_2<+1$. Therefore we have

$$(1-r)\left(\frac{a}{1-r^2}\right)=S_n.$$

Then, finally:

$$S_n=\frac{a(1-r)}{(1-r)(1+r)}=\frac{a}{1+r}.$$

But we know that $r=-k$, therefore we can at last write,

$$S_n=\frac{a}{1-k}$$

which means that even for this case $(-1<k<0)$ the formula $a/1-k$ as established by algebra is perfectly all right and indicates an exact value and not an approximate one.

DIRAN SARAFYAN
Graduate Student
Teachers College
Columbia University

[5] *Editor's Note:* This grouping of terms in an infinite series is dangerous and requires care, witness the three different answers one gets from considering $1-1+1-1+1-1\cdots+(-1)^{n-1}\cdots$, and $(1-1)+(1-1)+(1-1)+\cdots$, and $1+(-1+1)+(-1+1)+\cdots$. Mr. Sarafyan's procedure *is* valid though he did not show it to be so. It may be useful pedagogically and hence we print it, but students should be warned of the pitfalls surrounding infinite processes.—P. S. J.

42. *Simultaneous Linear Equations*

To find how long it takes A traveling 30 miles per hour to overtake B traveling 24 miles an hour with a two hour start, we let,

t = number of hours A drives,
$t+2$ = number of hours B drives,
$30t$ = number of miles A drives,
$24(t+2)$ = number of miles B drives.

The distances covered by A and B, from hour to hour, are entered in the second and third columns, respectively, of the following table:

t	$30t$	$24(t+2)$	$48-6t$
0	0	48	48
1	30	72	42
2	60	96	36
3	90	120	30

FIG. 5

The fourth column gives B's lead over A since $24(t+2)-30t=48-6t$. Equivalently stated,

A's distance + B's lead = B's distance.

An obvious check, therefore, is to note whether the entries of the second and fourth columns add up to that of the third column for $t=0, 1, 2, 3$.

Since in Figure 6 the connected segments PQ, QR, RS, have the same

$$\text{slope}=30/1=30,$$

the first set of points lie on a straight line whose slope is 30. Similarly the second

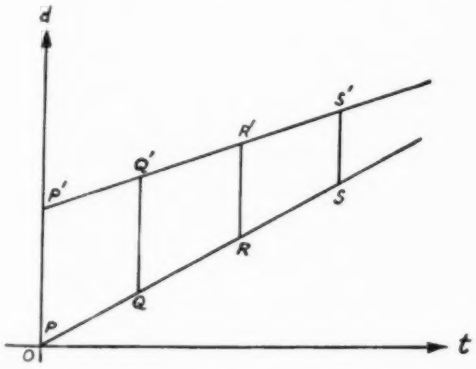


FIG. 6

set P', Q', R', S' lie on a straight line whose slope is 24. These straight lines are therefore the graphs of the *linear* functions,

$$\begin{aligned} d&=30t \\ d&=24t+48. \end{aligned} \qquad (1)$$

Since the segments from the t-axis to Q and Q', are the A and B distances, respectively, for $t=1$, their difference QQ' is B's lead as of that time. In like manner PP', RR', SS', represent B's lead for $t=0$, 2, 3. The decreasing entries of column 4, Figure 5, may be viewed, therefore, as line segments which shorten uniformly to the right. If we move these segments downward so that their lower ends rest on the t-axis, we obtain the graph of the *linear* function $48-6t$.

The slope of B's graph, 24, is less than A's, 30. The two lines must therefore meet if produced far enough to the right. At the point of intersection B's lead shrinks to zero and thus,

$$48-6t=0, \qquad (2)$$

whence

$$t=8,$$

and substituting in (1),

$$d=240.$$

We could also argue that the graphs meet where (when) the A and B distances are equal. Hence,

$$24t+48=30t, \qquad (3)$$

which reduces to equation (2).

Confirming the key role played by the supernumerary $48-6t$, and illustrating the fact that a linear function lends itself to arithmetic reasoning, we note that since B's lead of 48 miles is decreased 6 miles hourly, 8 hours suffice to reduce it to zero. Arithmetic thinking is vital to the understanding of algebra and should be encouraged at every step. It must however be tied in with algebraic, i.e., literal number, procedure. Thus the student should see B's lead as the decreasing sequence 48, 42, 36, 30, $\cdots$, but at the same time he must realize that these values result from $48-6t$ for $t=0$, 1, 2, 3.

Moreover the student should observe that

$$30t-24t=6t=t(30-24)$$

That is to say, since A gains $30t$ and B, $24t$, miles in t hours, A gains $6t$ miles over B. Taking the factored form of the left member we can argue equivalently that A gains $30-24=6$, miles an hour and

hence $6t$ miles in t hours. Subtracting A's gain of $6t$ from B's initial 48, leaves $48-6t$ as B's lead after t hours. Subtracting A's distance from B's distance i.e. $30t$ from $24t+48$ yields $48-6t$ in one step.

Stated abstractly our problem has been to find a simultaneous solution of equations (1). In general, as the graphs clearly show, the values of $30t$ and $24t+48$, for a given value of t, are different. It is, however, also evident, and we shall prove it in the concluding paragraph, that one, and only one t exists for which the two d's are equal. To find this t we write equation (3) or equivalently, equation (2).

In the general case of two first degree equations in two variables

$$\begin{aligned} px+qy&=r,\\ p'x+q'y&=r', \end{aligned} \qquad (4)$$

we can solve for $y(q,\ q'\neq 0)$, in the form,

$$\begin{aligned} y&=mx+b,\\ y&=m'x+b', \end{aligned} \qquad (4)'$$

and think in terms of an overtaking problem in which A and B travel m and m' miles per hour with starting distances of b and b' miles, respectively.

Equating B's (or A's) lead to zero,

$$(mx+b)-(m'x+b')=0, \qquad (5)$$

whence

$$x=(b-b')/(m'-m), \quad (m'\neq m) \qquad (6)$$

and substituting in either of equations (4)',

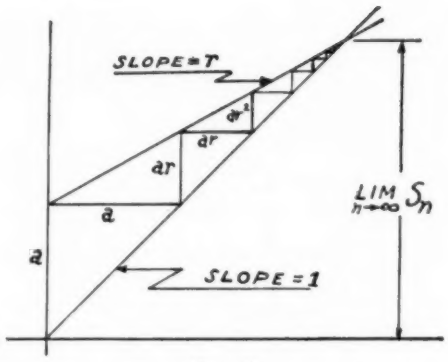


FIG. 7

$$y=(bm'-b'm)/(m'-m). \qquad (7)$$

The condition $m'\neq m$, rules out the possibility that A and B travel at the same rate, or what is the same thing, that their graphs are parallel lines.

A. R. JERBERT
University of Washington
Seattle, Washington.

Editor's Note: This discussion also suggests that more attention could be paid profitably, in secondary teaching, to the consideration of the utility of and the techniques for the direct graphical solution of problems.

Further, the reader should note the connections of the above note with the preceding note on geometric progressions. Any convergent infinite geometric progression may be summed graphically as the vertical distance to the intersection of a line through the origin with slope $=1$ and the line with y intercept of a and slope of r, the common ratio of the progression, as in Figure 7.

This may be verified by reasoning directly from the drawing or by writing the equations of the two lines and solving them simultaneously. P. S. J.

HAVE YOU SEEN?

"Binary Numeration Before Leibniz" by John W. Shirley in *American Journal of Physics*, November, 1951. "Though it is frequently stated that binary numeration was first formally proposed by Leibniz as an illustration of his dualistic philosophy, the mathematical papers of Thomas Hariot (1560–1621) show clearly that Hariot not only experimented with number systems, but also understood clearly the theory and practice of binary numeration nearly a century before Leibniz's time."

Also in the same issue:

"C.V. Boys' Rainbow Cup and Experiments with Thin Films" by John Satterly.

"The Forms of Cartesian Ovals in an Optical Range" by H. W. Farwell.

"On Aerophysics Research" by R. J. Seeger.

"Scientists and Mobilization."

"Scholarships and Fellowships Available at Institutions of Higher Education" by Theresa Birch Wilkins has been announced as the title of Office of Education Bulletin 1951, No. 16. This 248 page report is priced at 55 cents and may be purchased from the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C.

AIDS TO TEACHING

Edited by

HENRY W. SYER
School of Education
Boston University
Boston, Massachusetts

and

DONOVAN A. JOHNSON
College of Education
University of Minnesota
Minneapolis, Minnesota

BOOKLETS

B. 95—Figuring Methods for Civil Engineering

B. 96—Methods for Algebra

Monroe Calculating Machine Company, Orange, N. J.

Booklets; Free

Description of B. 95: This booklet ($7\frac{1}{2} \times 10\frac{1}{2}''$, 63 pages) gives definite steps to show how many engineering problems would be solved by machine methods. The following topics are covered: course and distance, open traverses, closed traverses, area, three-point problem, end areas and vertical curves.

Description of B. 96: This booklet ($8\frac{1}{2}'' \times 11''$, 21 pages) gives definite steps to determine the numerical value of 21 very common algebraic expressions. Since the solutions of most problems can be worked down to one of these forms on paper, this booklet should be a very helpful guide to transfer the numerical work to the machine.

Appraisal of B. 95 and B. 96: Since advanced machine computation is not the work of many mathematics courses these books will not be used in their entirety. However, the studying and understanding of even one problem from either book will convince pupils that the fundamentals of mathematics which they are learning run on and on through the applications which follow. The complicated numbers which are used are a revelation to many people who do not realize the complexity of the arithmetic found in many problems where the theory is simple.

CHARTS

C. 28—History of 10,000 Life Insurance Policy Holders

Institute of Life Insurance, 488 Madison Avenue, New York 22, N. Y.

Chart; $35'' \times 37''$; $0.15

Description of C. 28: In bright colors this chart shows the number of people living, the total premiums paid on $1000 ordinary life insurance policies, the total interest earned on investments, the number of persons dying, the total of death benefits paid, and the fund at the end of each ten year period from 20 years old to 100 years old.

Appraisal of C. 28: This is a factual and worthwhile chart for every mathematics classroom to have. Junior high pupils will begin to make simple mathematical comparisons (differences and ratios), to find maximum values and to see the need for graphical presentation and its advantages. Senior high pupils can work their way to considerations of probabilities. There is no doubt that an ingenious teacher can derive a great deal of mathematics from this chart.

FILMSTRIPS

FS. 95—Addition and Subtraction Concepts

Eye Gate House, 330 W. 42d Street, New York, N. Y.

Color ($4.00); 25 frames; one of 9 filmstrips in "Work and Play with Numbers" series. The price for the complete set is $25.00.

Description: The filmstrip introduces all the possible addition concepts for combin-

ing two numbers to get a sum of five and the subtraction of numbers from numbers equal to or less than five. In developing these concepts pictures of the following familiar objects are used: balloons, blocks, chicks, bears, dogs, birds, crows, pigs, toys, ducks. One frame shows a mother bear and two baby bears and explains that two plus one gives three. Another shows two boys and two girls and asks, "How many children do you see?" A third frame shows two pigs inside a pen and two pigs outside the pen and asks how many pigs are inside and how many are outside the pen. It is then explained that four take away two leaves two. The last frame reviews the various addition and subtraction concepts which have been developed.

Appraisal: (Based upon evaluation sheets prepared by a group of forty-two mathematics teachers.) This filmstrip is of most value for the following: to provide a common experience; to augment explanations; and to motivate. All the evaluators felt that this filmstrip is best suited for use in teaching primary grades arithmetic. A majority of the evaluators agreed with each of the following: everyday situations are used to illustrate the applications of mathematics; not too many ideas presented per frame; would hold the interest of the student; the filmstrip would encourage further pupil activity in mathematics and in other subjects; the vocabulary is suited to the primary grades level; the material is presented in a logical sequence; and the filmstrip attempts to supplement rather than replace the teacher. A majority of the evaluators felt, however, that the content can be just as, or more, effectively and efficiently presented in some other way. When asked whether or not they would advise their schools to purchase this filmstrip the evaluators responded as follows: 20, yes; 19, no; 3, did not answer. (Reviewed by Alton W. Clark, Kennebunk, Me.)

FS. 96—Arithmetical Concepts

Eye Gate House, 330 W. 42d Street, New York, N. Y.

Color ($4.00); 25 frames; from "Work and Play with Numbers" series.

Description: This filmstrip presents, through the use of pictures of objects familiar to children in the lower grades, the meanings of many words which are used in the study of arithmetic. One frame shows three children with different sized balloons and asks who has the biggest and littlest one. Another shows a car, a bicycle, and a plane and asks which moves fastest and which moves slowest. Still another frame pictures a block, a pencil, a piece of paper and a ball; explaining that the block is square, the pencil is long, the paper is flat, and the ball is round. Other words used in the study of arithmetic which are presented are as follows: big, little, taller, shorter, most, least, thick, thin, fastest, slowest, full, half, empty, higher, lower, heavier, straight, high, low, many, whole, near, far away, first, second, third, fourth.

Appraisal: (Based upon evaluation sheets prepared by a group of forty-two mathematics teachers.) This filmstrip is best suited to provide a common experience, to review, and to motivate. All evaluators felt that the filmstrip attempts to supplement rather than replace the teacher, and most agreed that the filmstrip is best suited for use in teaching primary grades arithmetic. A majority of the evaluators agreed with each of the following: everyday situations are used to illustrate the applications of mathematics; not too many ideas presented per frame; would hold the interest of the students; the filmstrip would encourage further pupil activity in mathematics, and in other subjects; the vocabulary is suited to the primary grades level; and the material is presented in a logical sequence. Thirty of the evaluators felt, however, that the content can be just as, or more, effectively and efficiently presented in some other way. When asked whether or not they would advise their schools to purchase this filmstrip, the evaluators responded as follows: 17, yes; 21, no; 4, did not answer. (Reviewed by Alton W. Clark.

FS. 97—We Learn Numbers, Part I

Eye Gate House, 330 W. 42 Street, New York, N. Y.

Color ($4.00); 25 frames; from "Work and Play with Numbers" series.

Description: The filmstrip introduces the numbers one through five. In the first frame there is a picture of one flower. The next frame shows a boy and a dog with the caption, "I have one dog. His name is Buster." In the following frames pictures are shown of other objects which are familiar to children, including the following: fish, chicks, cows, trees, dolls, bears, dogs, marbles. In the final frame devoted to the presentation of each new number it is shown how the number is written. The last few frames are used to review, testing the pupil's abilities to recognize the number of objects presented in each frame.

Appraisal: (Based upon the evaluation sheets prepared by a group of forty-two mathematics teachers.) There was complete agreement by all evaluators that everyday situations are used to illustrate the applications of mathematics, that the filmstrip would hold the student's interest, and that this filmstrip is best suited for teaching primary grades arithmetic. The group felt that the three greatest values of the filmstrip are as follows: to motivate; to review; and to augment explanations. A majority of the evaluators agreed with each of the following: there were no mathematical inaccuracies; not too many ideas presented per frame; the teaching methods used are conducive to learning; the filmstrip would encourage further pupil activity in mathematics, and in other subjects; the vocabulary is suited to the level of the primary grades; the material is presented in a logical sequence; and the filmstrip attempts to supplement rather than replace the teacher. On the other hand about three-fourths of the evaluators felt that the content can be just as, or more, effectively and efficiently presented in some other way. When asked if they would advise their schools to purchase this filmstrip, half the group answered yes, and half answered no. (Reviewed by Alton W. Clark, Kennebunk, Me.)

FS. 98—We Learn Numbers, Part II

Eye Gate House, 330 W. 42d Street, New York, N. Y.

Color ($4.00); 25 frames; from "Work and Play with Numbers" series.

Description: This filmstrip is a continuation of *We Learn Numbers, Part I*, and presents the numbers from six through ten. In the various frames, six, seven, eight, nine or ten common objects are shown. One frame pictures seven flowers with the caption, "See the seven flowers." Another frame shows a girl and eight baby chicks with the caption, "The girl is feeding eight chicks." In other frames the following common objects are shown: birds, apples, candles, clowns, books, pigs, dresses, stars, pennies. The final few frames provide a review in recognizing the number of objects presented. This review covers all numbers from one through ten.

Appraisal: (Based upon evaluation sheets prepared by a group of forty mathematics teachers.) This filmstrip is of most value for the following: to review; to provide a common experience; and to motivate. All the evaluators felt that this filmstrip is best suited for use in teaching primary grades arithmetic. All the evaluators also indicated that they thought the filmstrip would hold the student's interest. A majority of the evaluators agreed with each of the following: everyday situations are used to illustrate the applications of mathematics; the mathematical content contains no inaccuracies; not too many ideas presented per frame; the filmstrip would encourage further pupil activity in mathematics, and in other subjects; the vocabulary is suited to the primary grades level; the material is presented in a logical sequence; and the filmstrip attempts to supplement rather than replace the teacher. A majority of the evaluators felt, however, that the content can be just as, or more, effectively and efficiently presented in some other way. In response to

the question, "Would you advise your school to purchase this filmstrip?" 19 answered yes, and 16 answered no. (Reviewed by Alton W. Clark, Kennebunk, Me.)

FS. 99—Work and Play with Numbers, 5 and 6

Eye Gate House, 330 W. 42d Street, New York, N. Y.

Color ($4.00); 25 frames; from "Work and Play with Numbers" series.

Description: In this filmstrip combinations of two numbers which can be added to give a total of five or six are presented. It is also shown how to subtract numbers from both five and six. In all frames pictures of objects familiar to children in the lower grades are used. The first frame shows three little ducks and two big ducks. After asking how many big ducks and little ducks there are, it is shown that three plus two is five. In another frame there are three children standing in a group and two children running away from the group. It is shown that five (total number of children) take away two (the two children running away from the group) leaves three. A third frame shows three ducks in water and three on the bank. It is explained that six take away three leaves three. Several frames at the end are devoted to a review of the concepts in adding and subtracting which have been presented.

Appraisal: (Based upon evaluation sheets prepared by forty mathematics teachers.) This filmstrip is of most value for the following: to provide a common experience; and to review. All the evaluators felt that this filmstrip is best fitted for use in teaching arithmetic in the primary grades. A majority of the evaluators agreed with each of the following: the filmstrip uses everyday situations to illustrate the applications of mathematics; the students' interest would be held; the filmstrip would encourage further pupil activity in mathematics and in other subjects; the vocabulary is suited to the level of the primary grades; and the filmstrip attempts to supplement rather than replace the teacher. On the other hand a majority of the evaluators felt that the teaching methods are only partially conducive to learning, that the mathematical content contains many inaccuracies, and that too many ideas were presented per frame. Five answered yes and thirty-four answered no when asked if they would recommend their schools to purchase this filmstrip. (Reviewed by Alton W. Clark, Kennebunk, Me.)

FS. 100—Work and Play with Numbers, 7 and 8

Eye Gate House, 330 W. 42d Street, New York, N. Y.

Color ($4.00); 25 frames; from "Work and Play with Numbers" series.

Description: Pictures of objects familiar to children in the lower grades of school are used to show the combinations of numbers which can be added to give seven or eight, and how to subtract numbers from seven and eight. A detailed description of a few representative frames will show how these concepts of addition and subtraction are developed. One frame shows a boy with five blue and two orange balloons. It is explained that five plus two gives seven. In another frame four birds are shown sitting and three others flying. It is stated that seven (the total number of birds) minus three (the number flying away) leaves four (the number of birds sitting). In a third frame there are a total of seven kittens shown. A boy is carrying two of them away from the rest of the group. It is explained that seven kittens, take away two kittens, leaves five kittens. At the end of the filmstrip there are several review frames covering the material which has been presented.

Appraisal: (Based upon the evaluation sheets prepared by forty-three mathematics teachers.) This filmstrip is of most value for the following: to review; to motivate; and to augment explanations. A majority of the evaluators agreed that this

filmstrip is best suited for teaching arithmetic in the primary grades. A majority of the evaluators also agreed with each of the following: the filmstrip makes use of everyday situations to illustrate the applications of mathematics; not too many ideas are presented per frame; the filmstrip would hold the interest of the students; the filmstrip would encourage further pupil activity in mathematics and in other subjects; the vocabulary is suited to the level of the primary grades; the material is presented in a logical sequence; and the filmstrip attempts to supplement rather than replace the teacher. When asked if they would advise their schools to purchase this filmstrip, four answered yes, and thirty-six answered no. Most of the evaluators felt that the material could be just as, or more, effectively and efficiently presented in some other way. (Reviewed by Alton W. Clark, Kennebunk, Me.)

FS. 101—Work and Play with Numbers, 8 and 9

Eye Gate House, 330 W. 42d Street, New York, N. Y.

Color ($4.00); 25 frames; from "Work and Play with Numbers" series.

Description: As in the previous filmstrips of this number series, various pictures of common objects are presented to develop the combinations of numbers which when added will give eight or nine. The subtraction of numbers from eight and nine is also presented. One frame shows seven white ducks and one brown duck; it is stated that seven plus one gives eight. In the following frame the brown duck has gone to the shore, eight ducks (total number of ducks) minus one duck (the one on shore) leaving seven ducks (those remaining in the water). Other frames show that six rabbits and three rabbits are nine rabbits, nine cows take away one leaves eight, and other addition and subtraction concepts. Several review frames to test knowledge of the material which has been presented are included.

Appraisal: (Based upon evaluation sheets prepared by a group of forty-three mathematics teachers.) This filmstrip is of most value to provide a common experience, to review, and to motivate. All but one evaluator agreed with each of the following: the filmstrip is best suited for teaching arithmetic in the primary grades; the filmstrip uses everyday situations to illustrate the applications of mathematics; and the filmstrip would hold the interest of the students. A majority of the evaluators also agreed with the following: not too many ideas are presented per frame; the filmstrip would encourage further pupil activity in mathematics and in other subjects; the vocabulary is suited to the primary grades level; the material is presented in a logical sequence; and the filmstrip attempts to supplement rather than replace the teacher. On the other hand a majority felt that the content can be just as, or more, effectively and efficiently presented in some other way, and that the teaching methods of the filmstrip are only partially conducive to learning. When asked if they would advise their schools to purchase this filmstrip, nine answered yes, and thirty answered no.

FS. 102—Work and Play with Numbers, 9 and 10

Eye Gate House, 330 W. 42d Street, New York, N. Y.

Color ($4.00); 25 frames; from "Work and Play with Numbers" series.

Description: This filmstrip presents the various combinations of two numbers which can be added to give totals of nine and ten. It also shows the numbers left when various numbers are subtracted from either nine or ten. As in the previous filmstrips of this series, pictures of objects which are familiar to children in the lower grades are used in developing the concepts. Objects pictured include the following: balloons, apples, flowers, marbles, birds, leaves, cents, pigs, cups. One frame pictures nine boys and one girl and explains that nine boys plus one girl gives ten children. Another shows eight birds sitting

and two birds flying, explaining that eight birds plus two birds gives ten birds and that ten birds (total number of birds) minus two birds (those flying away) leaves eight birds (those sitting). Several frames at the end of the filmstrip are reserved to review the material which has been presented.

Appraisal: (Based upon evaluation sheets prepared by a group of forty-three mathematics teachers.) This filmstrip is of most value for the following: to provide a common experience; to motivate; and to review. All the evaluators agreed that the filmstrip is best suited for teaching arithmetic in the primary grades, and that it would hold the students' interest. A majority of the evaluators agreed with each of the following: the filmstrip uses everyday situations to illustrate the applications of mathematics; not too many ideas are presented per frame; the filmstrip would encourage further pupil activity in mathematics, and in other subjects; the vocabulary is suited to the level of the primary grades; the material is presented in a logical sequence; and the filmstrip attempts to supplement rather than replace the teacher. On the other hand a majority also felt that the content can be just, or more, effectively and efficiently presented in some other way, and that the teaching methods of the filmstrip are only partially conducive to learning. Ten teachers would and twenty-eight teachers would not advise their schools to purchase this filmstrip. (Reviewed by Alton W. Clark, Kennebunk, Me.)

FS. 103—Time and Money

Eye Gate House, 330 W. 42d Street, New York, N. Y.

Color ($4.00); 25 frames; from "Work and Play with Numbers" series.

Description: The purposes of this filmstrip are to teach pupils to identify the positions of the hands of a clock at the various hours, and to understand the relationships between pennies, nickels and dimes. The first twelve frames show the positions of the hands of a clock for each hour, at random rather than chronological order. In the presentation of money a picture of first one cent and then of five pennies is shown. A picture of a nickel is shown and it is explained that a nickel is the same as five pennies. Next a picture of ten pennies is shown, followed by a picture of a dime and the explanation that a dime is the same as ten pennies. It is next shown that a dime is the same as two nickels.

Appraisal: (Based upon evaluation sheets prepared by a group of thirty-nine mathematics teachers.) This filmstrip is of most value for the following: to review; to motivate; and to provide a common experience. All the evaluators agreed that this filmstrip is best suited for use in teaching arithmetic in the primary grades. A majority also agreed with each of the following: the filmstrip uses everyday situations to illustrate the applications of mathematics; the mathematical content contains no inaccuracies; the filmstrip would hold the interest of the students; the vocabulary is suited to the level of the primary grades; the material is presented in a logical sequence; and the filmstrip attempts to supplement rather than replace the teacher. On the other hand a majority of the evaluators felt that the content can be just, or more, effectively and efficiently presented in some other way, and that the teaching methods used are only partially conducive to learning. Seven answered yes and thirty-one answered no when asked if they would advise their schools to purchase this filmstrip. (Reviewed by Alton W. Clark, Kennebunk, Me.)

Change in Price: C.23 Mathematics Calendar (March, 1951) available from Miss Laura E. Christman is now priced at 65¢ each and not 50¢.

APPLICATIONS

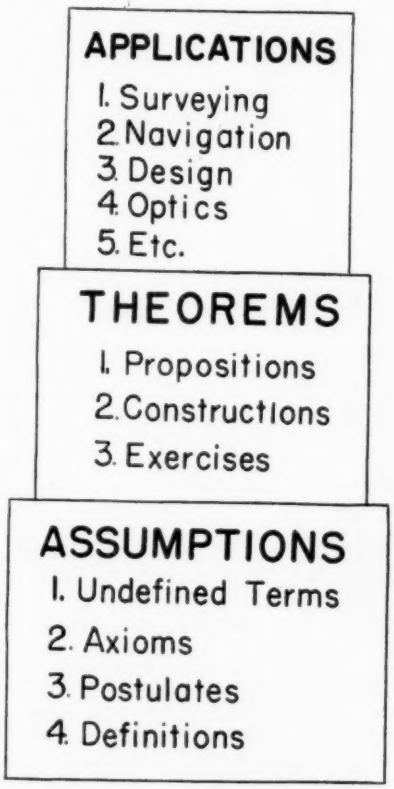
Edited by SHELDON S. MYERS
Department of Education, Ohio State University, Columbus, Ohio

You will recall that last month's department presented some military applications and references. An additional reference on military mathematics appropriate to the secondary school has been brought to our attention entitled *Military Applications of Mathematics* by Paul P. Hanson, McGraw-Hill Book Company, New York, 1944, 447 pages.

P. 5 Gr. 10–12 *Two Definitions of Importance to a Community*

Many geometry teachers do not appreciate the general importance of definitions in both geometric and non-geometric situations because they do not fully understand the structure of all postulational systems, in other words, the nature of proof. *Thinking About Thinking* by Cassius J. Keyser is one of the best little volumes (readable in about an hour) on the nature of proof for both high school students and high school teachers. The structure of the postulational method is exhibited in such diverse systems as quantum mechanics, the law, philosophic systems, Euclidean and non-Euclidean geometries, higher algebra, symbolic logic, and the calculus of classes and the calculus of propositions. "Rigor" in mathematics usually means adherence strictly to the postulational method. The last fifty years has been characterized by the extensive development of the postulational method in both mathematics and the sciences, especially physics.

In the new approach to plane geometry, called "The Nature of Proof," I have usually presented the outline of the structure of postulational systems with the following drawing called "Pillar of Proof."



PILLAR OF PROOF

The full meaning of this pillar is brought out by the entire course. Reference is made to it at appropriate times throughout the year as the students erect their own geometric edifice. All assumptions, including definitions, are group agreements made after adequate group discussion. The essential theorems are discovered by students inductively and built into their system by deduction. Students will vary in the number of additional theorems they will discover and establish, and in the kind and extent of the applications

they will make of them. The entire geometric edifice of each student is presented in a highly organized notebook which he composes as he proceeds.

I have felt the above remarks would make more significant two illustrations of the importance of definitions by HOWARD E. ROBINSON of Piqua High School, Piqua, Ohio. He writes: "As a member of the N.C.T.M. and as an ardent reader of its magazine, I am enclosing two examples which I have found and used. Perhaps they may be of some use to other readers." (He referred to P. 3 Gr. 10–12 in the October, 1951 issue.)

From the North Canton, Ohio, paper:

> Due to the many varieties of things which have been appearing in the garbage cans it has become necessary for the Village Council to define the word "Garbage" as meaning the offal of all food originally intended for human consumption, and it does NOT include paper, tin cans, glass, cardboard, corrugated boxes, and other food containers.

From the East Liverpool, Ohio, Review:

> "The Police Department has been instructed to crack down on delivery truck drivers who take advantage of a lenient policy on parking for loading and unloading merchandise," the Safety-Service Director said today.
>
> Traffic checkers will continue to show leniency toward truck drivers who stop by a parking meter for a short time without depositing a coin, but those who "take unreasonable advantage" of the policy will be ticketed, the director warned.

Nature of proof classes or geometry classes stressing proof could investigate the interpretation of the two definitions, the scope of them, possible exceptions and difficulties of implementation, and whether the terms used in each definition are semantically clear and less complex than the word being defined. Similar investigations should be made with geometric definitions. The principle of reversibility could be applied on both geometric and non-geometric definitions.

Al. 10 Gr. 9–12 *If a Cricket Chirps One Hundred Times a Minute, How Fast Does an Ant Run?*

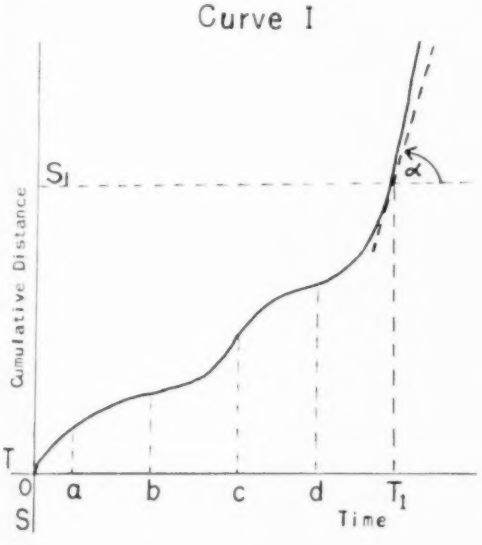
Teachers and students everywhere should get a kick out of the above bit of wit and functionality sent to me by WALTER H. CARNAHAN of Purdue University. You won't believe it at first, but the question has the computable answer of 2 3/11 inches per minute! The following explanation by Carnahan should clear up your perplexity and enable even your ninth graders to compute the answer by simple algebra.

"To be a little more precise, I should specify a *common* cricket and the line-running ant called *Tapinoma sessile*, but that becomes a bit technical. Lutz in *A Lot About Insects* points out that observation indicates that the Fahrenheit temperature equals the number of chirps of the common black cricket in 15 seconds plus 39, or $T=C+39$. A study by Harlow Shapley of the *Tapinoma sessile* shows that, in inches per minute, the ant's speed and the Fahrenheit temperature are related by the formula $T=11S+39$. Hence, the problem, and lo, the answer!"

Calculus 1 Gr. 12–14 *The Meaning of Derivative and Integral Illustrated by the Speedometer*

The common, everyday device known as the speedometer constantly illustrates the derivative and integral of calculus. The teaching of an elective course in calculus for the twelfth grade was recommended by the National Committee of 1923. One of the ardent proponents of the teaching of calculus in high school was the late John A. Swenson who wrote the chapter, "Selected Topics in Calculus for the High School," in the Third Yearbook of The National Council of Teachers of Mathematics. The many teachers today who are teaching the calculus to seniors are anxious to find simple ways to illustrate the meaning of derivative and integral within the experience of students. The following example occurred to me and has proved successful in clarifying the meaning of derivative and integral for high school seniors.

The speedometer is a very familiar instrument which illustrates both the derivative and the integral. The part of the instrument which shows the speed at any given moment is really giving the derivative continuously of the curve plotting cumulative distance against the time for this particular automobile. The part of the instrument which records the total trip is really giving the integral of the curve plotting speed against time for this particular automobile. These two curves with the derivative of one and the integral of the other can be compared by the graphs below:



Ordinate is Total Trip.

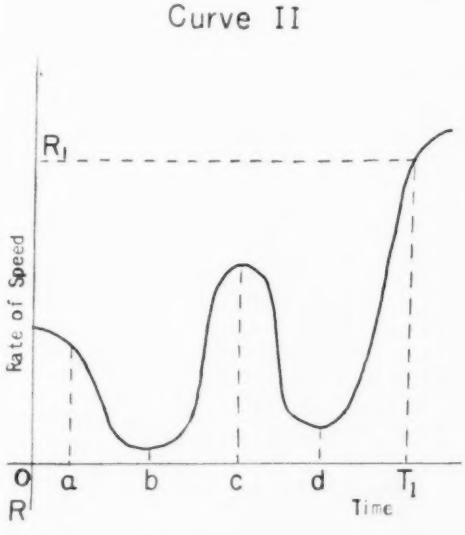
If $S = f(T)$, then

$\frac{dS}{dT}$ = derivative of $f(T)$

$= f_1(T) = R.$

If $T = T_1$, then $\frac{dS}{dT} = R_1 = f_1(T_1)$

or $\tan\alpha = R_1$.



Ordinate is Speed.

If $R = f_2(T)$, then

$\int_0^{T_1} R\,dT = \int_0^{T_1} f_2(T)\,dT = S_1$

$=$ total trip.

S_1 = total trip = area under curve between $T = 0$ and $T = T_1$.

These two curves are interesting in that the integral of Curve II is the ordinate of Curve I, while the derivative of Curve I is the ordinate of Curve II. Furthermore, the linear portions of Curve I are the horizontal portions of Curve II. The steep portions of Curve I are high portions of Curve II, while the more horizontal portions of Curve I are the lower portions of Curve II. Check this by comparing the corresponding times a, b, c, d, and T_1 on both curves. Although this illustration may not be appropriate at the very beginning of the study of calculus, there is a stage, to be determined by the situation, when this illustration will greatly strengthen understanding.

Al. 11 Gr. 9–12 *Developing Understanding of $Y = mX + b$ and of the Simultaneous*

Graphical Solution of a Problem in Two Unknowns.

Problem: Rectangle (1) in the figure below starts with an area of 3 and rectangle (2) starts with an area of 6. Both are allowed to increase their lengths. By how

(Continued on page 63)

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Department of Education, Brooklyn College, Brooklyn, N. Y.

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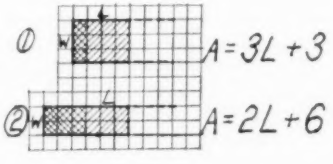
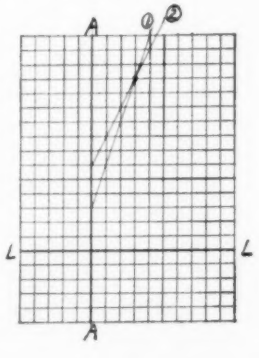
Correction: In bibliography for "Laboratory Mathematics" in the October, 1951 issue, page 422, under heading 2, "Laboratory Devices for Plane Geometry" the author of "A Multiple Purpose Device" should be listed as Eddy, Louise B. (Mrs. John M.) instead of Berger, E. J.

Applications

(Continued from page 59)

much must the lengths of each be increased so that they will both have the same area?





Note that the "*b*'s" of both equations are the ordinates or *Y*-intercepts and have the physical meaning of representing the initial starting areas of both rectangles. Observe also that the "*m*'s" of both equations are the slopes of each graph and represent the rate of increase of area with respect to length. These rates are determined by the respective widths of each rectangle. In rectangle (1) the area increases 3 units for each unit increase in length, while in rectangle (2) the area increases 2 units for each unit increase in length. Students should note this, both on the figure and on the curve. It can then be seen that, although curve (1) starts lower, its slope is greater than (2) and catches up with (2) when $l=3$. At this point on the graph, the two areas are equal. The analogous point in the figures is seen when rectangle (1) has the dimensions 3 by 4 and rectangle (2) has the dimensions 2 by 6, making both areas 12. In my opinion teachers who teach only the algebraic solution of this kind of problem are neglecting its most important aspect, the function concept.

The Third Delegate Assembly

JOHN R. MAYOR, *Chairman*
Committee on Affiliated Groups
The University of Wisconsin, Madison, Wisconsin

THE SESSION of the Third Delegate Assembly of the National Council of Teachers of Mathematics will be held in Des Moines, Iowa, Thursday and Friday, April 17 and 18, 1952. There will be a two-hour session on Thursday and a luncheon for Delegates followed by a second session of the Assembly on Friday. The agenda for the Assembly is to be made up from suggestions of officers of Affiliated Groups and other members of the National Council.

Questions on relationships between the Affiliated Groups and the National Council have been discussed at the first two Delegate Assemblies and a pattern of working relationships has now been established. This year the Assembly will consider problems of a broader nature which are of concern to all members of the National Council and of importance for the welfare of our organization and of mathematics teachers everywhere.

The Assembly serves in an advisory capacity to the Board of Directors and its actions are in the form of recommendations to the Board of Directors. In some instances, in the questions below, the Delegate Assembly will provide an opportunity for National Council officers responsible for leadership of the activity under consideration to obtain a sampling of opinion of Council members. Particularly on these questions the Delegates should seek to determine the point of view of members of the Groups they represent, so that the sample can be representative of as large a number as possible of those interested in the improvement of the teaching of mathematics.

Where a question refers to a particular service or agency of the Council those who are leaders in this work will be asked to report on the activity and lead the discussion of the Delegates. For example, when THE MATHEMATICS TEACHER is discussed the editor, E. H. C. Hildebrandt, will be present to take part in the consideration of this major service of the Council; and when year books are discussed, Veryl Schult, chairman of the Yearbook Committee, will present some pertinent problems related to the work of her committee.

The following list of questions has been prepared from suggestions of the Second Delegate Assembly, the Affiliated Groups, and others active in the work of the Council. The agenda of the Third Delegate Assembly will be made up from this list. All readers of THE MATHEMATICS TEACHER are invited to comment on these plans and make proposals for their development so that the Third Delegate Assembly can be an effective means of providing more active participation of members everywhere in the growth and progress of The National Council of Teachers of Mathematics.

Topics proposed for discussion in Des Moines and on which the Delegates may wish to make recommendations to the Board of Directors are indicated by the following questions:

1. Should we continue to have a section devoted to the work of Affiliated Groups at each meeting of the National Council? If so, what should be the nature of the program in these sections?
2. Should we continue to have at National Council meetings, sections sponsored, planned and carried out by Affiliated Groups in the geographical area of the host city?
3. What needs to be done to promote and improve special activities, initiated through the Affiliated Groups program?
4. What obligations should an Affiliated Group which is host organization for a National Council meeting be asked to assume?

5. How can the program organization for National Council meetings be improved?
6. What improvements or changes in THE MATHEMATICS TEACHER should be considered?
7. What is the value of the Yearbooks to teachers? To what topics should future Yearbooks be devoted?
8. What kind of small publications are needed?
9. How can the growth of the National Council best be promoted?
10. Can the Affiliated Groups assist in any way in bringing about the appointment of a mathematics specialist to the staff of the U. S. Office of Education?
11. Should an honorary membership be established for retiring teachers who have supported the National Council over a long period of years?
12. What means of determining problems for discussion and what procedures in the Delegate Assembly will serve the National Council most effectively?

NEWS NOTES

The Third Annual Institute for Teachers of Mathematics sponsored by The Association of Teachers of Mathematics in New England was held at Connecticut College, New London, Connecticut, August 23–29, 1951. The Institute was a great success, greeted with enthusiasm by all in attendance. One hundred seventy-five teachers from fourteen states, Washington, D. C., Ontario, and Beirut, Lebanon, joined in a program of cooperative activity.

A unique feature of this Institute is that it is sponsored by a group of mathematics teachers, who make all arrangements, including the choice of a different site each year. The preceding Institutes were held at Wellesley College in 1949 and at Tufts College in 1950. The Institute was started through the initiative of Professor Henry W. Syer of Boston University.

The formal program of the Institute comprises lectures, study groups, and a mathematics laboratory, and there is also opportunity for much informal discussion. The spirit is one of congeniality and good fellowship. The fact that about one-third of the members of the 1951 Institute have attended for the three consecutive years testifies to the enjoyment and profit derived from this pleasant association.

Lecturers at this year's Institute included: Professor Fred L. Strodtbeck, Yale University, "The Quantitative Study of Interaction in Small Groups"; Professor John T. Rule, Massachusetts Institute of Technology, "Graphics: the Neglected Mathematical World"; Edmund C. Berkeley, Consultant and Actuary, "Symbolic Logic and Applications"; Colonel Robert C. Yates, United States Military Academy, West Point, "Models and Methods"; Martha Hildebrandt, Proviso Township High School, Maywood, Illinois, "The Dilemma of the High School Mathematics Teacher"; Professor H. W. Brinkmann, Swarthmore College, "The Foundations of Arithmetic"; Professor Harlow Shapley, Harvard University, "Galaxies"; and Professor A. Day Bradley, Hunter College, "Map Projections."

Study groups on a variety of topics provided opportunity for much stimulating discussion. The groups and their leaders were: "Calculus for Teachers of Twelfth Grade Mathematics," Professor Edward M. Pease, Rhode Island State College; "Arithmetic and Junior High School Mathematics," Professor Ben A. Sueltz, State University College of Education, Cortland, New York; "Modern Mathematics after Tenth Grade," Professor Eugene P. Northrop, University of Chicago; "The Status of Mathematical Concepts and Skills at the End of the Sixth Grade," Professor Kenneth G. Fuller, Teachers College of Connecticut, New Britain, Connecticut; "Mathematics Clubs and Recreations," Professor E. H. C. Hildebrandt, Northwestern University and Editor of THE MATHEMATICS TEACHER; "A New Program in Grades Ten and Eleven," Professor Virgil S. Mallory, State Teachers College, Montclair, New Jersey; "The Place of Postulates in College Freshman Courses," Professor Albert A. Bennett, Brown University; "Cartesian Instead of Euclidean Geometry," Professor Julia Wells Bower, Connecticut College; "Mathematics Clubs in High School and College," Professor William R. Ransom, Tufts College; and a four-fold group on "How I Teach," led by Martha Hildebrandt on "Pupil Participation," Jackson B. Adkins, Phillips Exeter Academy, Exeter, New Hampshire, on "The Function Concept," Daniel B. Lloyd, Wilson Teachers College, Washington, D. C., on "For Appreciation," and George F. Porter, The Choate School, Wallingford, Connecticut, on "The Concept of Locus."

One of the busiest and happiest Institute groups could be found working in the Mathematics Laboratory, where a variety of teaching aids were constructed under the guidance of Allene Archer, Thomas Jefferson High School, Richmond, Virginia; Gladys Schuder, Lane

Members of the 1951 New England Mathematics Institute

Row I. Ransom, Tufts Col.; Mallory, Montclair, N. J.; Lawton, Fairhaven, Mass.; Schuder, Charlottesville, Va.; Bower, Conn. Col.; Sueltz, Cortland, N. Y.; O'Brien, Portland, Me.; Adkins, Exeter, N. H.; Little, Boston; Mergendahl, Newton, Mass.; Caruso, Milford, Conn.; Garland, Boston; Cochran, Somerville, Mass.; Bishop, Cambridge, Mass.; Archer, Richmond, Va.; Rice, Worcester, Mass.; Lloyd, Washington, D. C.; Bennett, Brown Univ.; M. Hildebrandt, Maywood, Ill.; E. Hildebrandt, Northwestern Univ.

Row II. Weishaupt, Bradford, Mass.; Black, Quincy, Mass.; E. Smith, Milford, Conn.; Chapin, E. Northfield, Mass.; Hazzard, Guilford, Conn.; Nourse, Rutland, Vt.; Hunt, Jamestown, N. Y.; Loring, Belmont, Mass.; Marcy, Bryn Mawr, Pa.; Campbell, Princeton, N. J.; G. Smith, Boston; Mrs. Garland, Boston; Mrs. Rice, Worcester, Mass.; Gray, New York City; Drees, Glens Falls, N. Y.; Pierce, Brockton, Mass.; Nicosia, Cliffside Park, N. J.; Cheney, Univ. of Conn.; Stoddard, Hingham, Mass.

Row III. Deck, Muhlenberg Col.; Watson, New London, Conn.; Howe, Mechanicville, N. Y.; Betts, Boston; McGraw, Lewiston, Me.; Carpenter, Barton, Vt.; White, Barre, Vt.; Truenfels, Beirut, Lebanon; Eddy, Providence, R. I.; Junkins, Hampton, N. H.; Sibley, Brooklyn, N. Y.; S. Curtis, E. Northfield, Mass.; Height, Wakefield, Mass.; Farnham, Portland, Me.; Hartnett, Ludlow, Mass.; Rowley, Taunton, Mass.; Podmele, Buffalo, N. Y.; Del Grande, Aurora, Ont.; Buell, Terryville, Conn.

Row IV. Finley, Windsor, Conn.; Mrs. Marsh, Boston; Macdonald, Fall River, Mass.; Miles, Fall River, Mass.; Thomas, Fall River, Mass.; Beattie, Fall River, Mass.; Martin, Fall River, Mass.; Locklin, Plymouth, Mass.; Dickerman, Naugatuck, Conn.; Selover, Cleveland, Ohio; Russell, Winsted, Conn.; Haskins, Fitchburg, Mass.; Huntington, Haverhill, Mass.; Clarke, Hackensack, N. J.; Bruce, Trenton, Ont.; Rice, Newington, Conn.; Mullaney, Bangor, Me.; Mrs. Stoddard, Hingham, Mass.; McCarthy, Peabody, Mass.; Nearing, Wallingford, Conn.

Row V. Richards, Far Rockaway, N. Y.; Mayer, Scarsdale, N. Y.; Witters, St. Albans, Vt.; Hurley, Bridgeport, Conn.; Hannay, Plainfield, N. J.; Curran, Univ. of Conn.; Young, White River Junc., Vt.; Bowen, Shaker Heights, Ohio; Collins, Providence, R. I.; C. Curtis, Chatham, Va.; Lougee, Newtonville, Mass.; Adamo, Cliffside Park, N. J.; Roper, Westport, Conn.; Green, Weston, Mass.; Minor, Woodstock, Conn.; Ormsby, Milton, Mass.; Br. Smith, Gates Mills, Ohio; Dobberstein, Bronxville, N. Y.

Row VI. Morley, Worcester, Mass.; Marsh, Boston; Austin, Cincinnati, Ohio; Storrs, Hartford, Conn.; Taylor, Quincy, Mass.; Reigh, State College, Pa.; Phillips, New York City; Keith, Schenectady, N. Y.; Roorda, Storm Lake, Iowa; Emerson, Haverhill, Mass.; Creamer, Holden, Mass.; Reid, Malden, Mass.; Speirs, Windsor, Conn.; Gitnig, Far Rockaway, N. Y.; Hughes, Highland Park, N. J.; Burtt, Plattsburg, N. Y.; R. Miller, Montpelier, Vt.; Heavenrich, Newton Center, Mass.; H. Miller, Granby, Conn.

High School, Charlottesville, Virginia, and Elwood M. Stoddard, High School, Hingham, Massachusetts.

Several rooms were given over to exhibits of aids to the teaching of mathematics, including textbooks and professional books, the collection of foreign books in possession of the Association, commercial charts and calculating devices, and models and devices made by teachers and students.

Interspersed with the more serious items of the program were many recreational activities. The spacious campus of Connecticut College,

(*Continued on page 70*)

DEVICES FOR A MATHEMATICS LABORATORY

Edited by EMIL J. BERGER
Monroe High School, St. Paul, Minnesota

This section is being published as an avenue through which teachers of mathematics can share favorite learning aids. Readers are invited to send in descriptions and drawings of devices which they have found particularly helpful in their teaching experience. Send all communications concerning **Devices for a Mathematics Laboratory** to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

A TEACHING DEVICE FOR GEOMETRY RELATED TO THE CIRCLE

The writer has found the teaching device suggested in this article very useful in illustrating theorems, ideas, and definitions related to the circle. No claim is made for originality and many teachers are already using it; however, the idea seems valuable enough to make passing it on worthwhile.

The device consists of a circle 15″ in diameter drawn on a piece of plywood. Evenly spaced around the circumference are twenty-four metal brads driven into the wood; a twenty-fifth brad is located at the center. In order to make the device look really finished the circle and background should be painted in contrasting colors. The only accessories needed to use the device are three or four loops of round elastic braid.

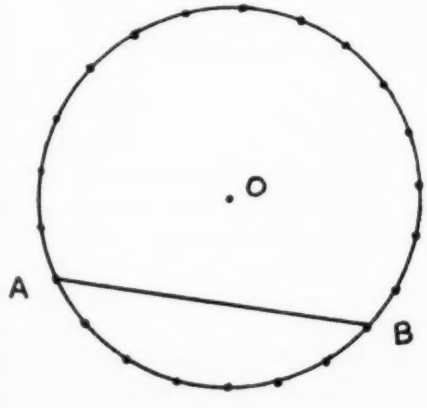


FIG. 1

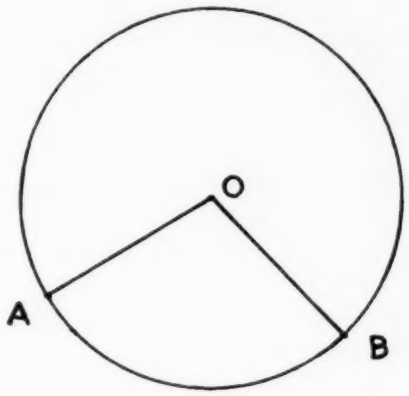


FIG. 2

Figures may be formed simply by looping the elastic from brad to brad. The accompanying figures suggest some of the possibilities. If a short loop is fastened around brads A and B as in Figure 1, a chord is formed. Experience with the device will show that the two strands of the loop lie so close together that they represent one line. In this position the loop around A and B may also be thought of as dividing the circle into two segments. If the two strands of the loop are stretched around the center O two central angles and two sectors are formed. (See Fig. 2.) By stretching the loop farther to the point C an inscribed angle is formed. (See Fig. 3.) Since the device is so constructed that there are approximately 15 arc degrees between two consecutive brads it is possible to compute the size of all inscribed angles that can be formed with the device and to check such computations with a protractor. By placing a second loop around A, D, and B, it is possible to illustrate two angles inscribed in the same

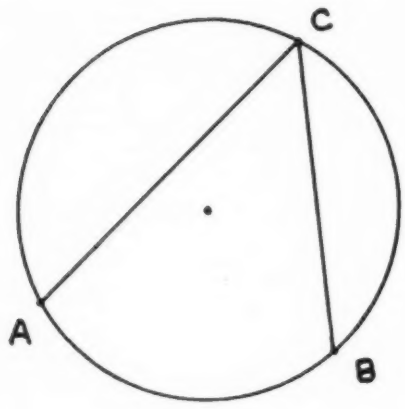


FIG. 3

segment. (See Fig. 4.) This arrangement is also suitable for proving theorems about intersecting chords. It will be noted that all necessary auxiliary construction lines are present. Actually Figure 4 can be produced with only one loop. This can be

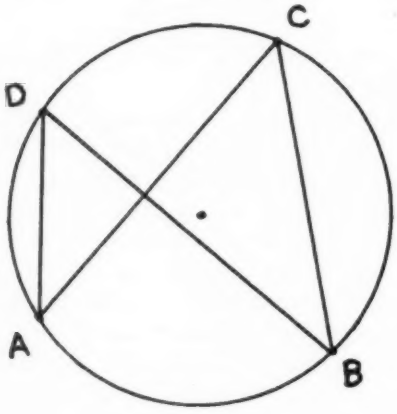


FIG. 4

done by placing the loop around A and D, twisting it once, and then stretching it around B and C.

An unlimited number of inscribed polygons can be formed including some of the regular polygons. Figure 5 illustrates the regular hexagon. Some teachers say that it is faster and more satisfactory to form figures with this device than to draw them on the blackboard. An interesting use of the device is in deriving the value of π. By forming in succession a square, hexagon, dodecagon, and finally a 24-gon the student can easily see that the polygon becomes more like a circle, with its perimeter approaching the circumference, and its area approaching the area of the circle, etc. The ingenuity of teachers and students will suggest other uses.

A mechanical teaching aid does more than assist the teacher with his explanations. The conclusions reached by using the aid should not of course be accepted

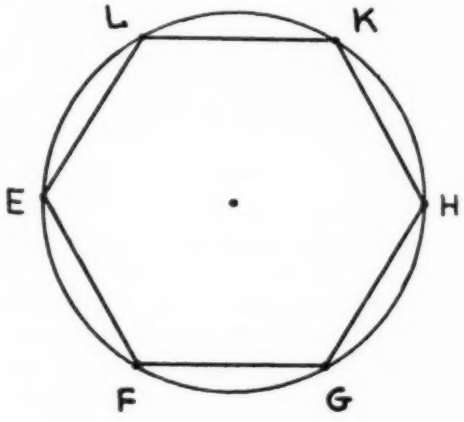


FIG. 5

as proved, but the use of the aid will help clarify ideas so that the final writing of the proof is made easier. Incidentally, if the teacher does not have time or the requisite mechanical ingenuity for constructing devices, often there are students in class who enjoy making them.

M. H. AHRENDT
Executive Secretary
National Council of Teachers
of Mathematics
Washington, D. C.

A PROJECTION DEVICE FOR FINDING THE AREA OF AN ELLIPSE

If a flashlight is held in a vertical position above a desk the rays of light emitted will, in general, be parallel to each other and strike the top of the desk perpendicularly. This is true because the reflector in the flashlight is a parabolic surface of revolution and the light bulb is at the focus of the surface.

Suppose now that the light is held about

4″ from the top of the desk and at the same time a penny is held midway between the light and the desk. If the coin is parallel to the plane of the desk, the shadow on the top of the desk will be circular and equal in area to the face of the coin. If the penny is tilted, the shadow will change to an ellipse. In either case the shadows may be thought of as projections of the coin upon the plane of the desk.

The problem of finding the area of an ellipse produced in the manner described above and the related problem of deriving a general formula for the area of any ellipse are usually studied in college mathematics classes. However, both problems can also be solved by high school trigonometry students. In fact both of these problems provide a nice application of the cosine function. The device described in the following paragraphs can be used as a model for the work.

The device itself is not very complicated. It consists essentially of a box with a slit top and open bottom and an arrangement for outlining the projection of a circle on the slit top. Convenient dimensions for the box are as follows: width, 8″; length, 14″; and depth, 12″. Apple-box lumber is adequate for constructing the entire device.

To build the slit top cut out fourteen strips of wood 1/2″×14″, sand all long edges smooth, and fasten the strips to the top of the box allowing a separation 1/16″ wide between adjacent strips.

Next cut out a circular disk 7″ in diameter and a strip of wood 3/4″×14.″ Nail one end of this strip along a diameter of the disk and fasten the other end to the edge of the box with a hinge as is shown in the diagram. (See Fig. 6.) Let the circular disk rest on the top of the box and locate the points on the circle which lie on the lines formed by the parallel slits. By using thumb tacks fasten strings about 12″ long to the edge of the disk at these points. Drop each string through the slit over which it is located and attach a fish line weight to the end of each as is illustrated by the artist's "cut-away" in the side of the box. Since the box has no bottom this will be a simple matter.

The intersections of the vertical strings and the plane of the slit top outline the projection of circle O' upon the slit top. By sawing into the edge of the box on one side of the hinge a metal protractor may be inserted. A short hat pin pushed into the end of the long wooden strip provides a tidy arrangement for measuring the angle of inclination of the circular disk with the plane of the slit top. It will be noted that the device is so designed that this angle (θ) may be varied as desired.

The projection of circle O' upon the plane of the slit top is an ellipse for all values of angle θ except 0° and 90°. The area S of the projection can be expressed as follows for all values of angle θ including 0° and 90°:

$$S = \pi(A'0')^2 \cos\theta.$$

If students question the truth of this formula they need only be shown that similar formulas can be developed for the projections of all plane figures of which a rectangle inclined to a plane is probably the simplest.

Following is a derivation of the formula for the area of any ellipse:

(1) In circle O', $A'O' = B'O'$.
(2) Area of circle $O' = \pi(A'O')^2$
$= \pi(A'O')(B'O')$.
(3) Draw $B'X \perp O'O$.
(4) Then $B'X = BO$, and angle $O'B'X$ = angle θ.
(5) Hence $BO = (B'O') \cos\theta$.
(6) From before, area of ellipse $O' = \pi(A'O')^2 \cos\theta$.
(7) Substituting in (6) from (2) and (5), and noting that $A'O' = AO$ we have the following result: area of ellipse $O' = \pi(AO)(BO)$.

Thus it is apparent that the area of any ellipse is π times the product of its semi-axes. The above derivation is certainly not unique. The department editor came across it a number of years ago.[1] However,

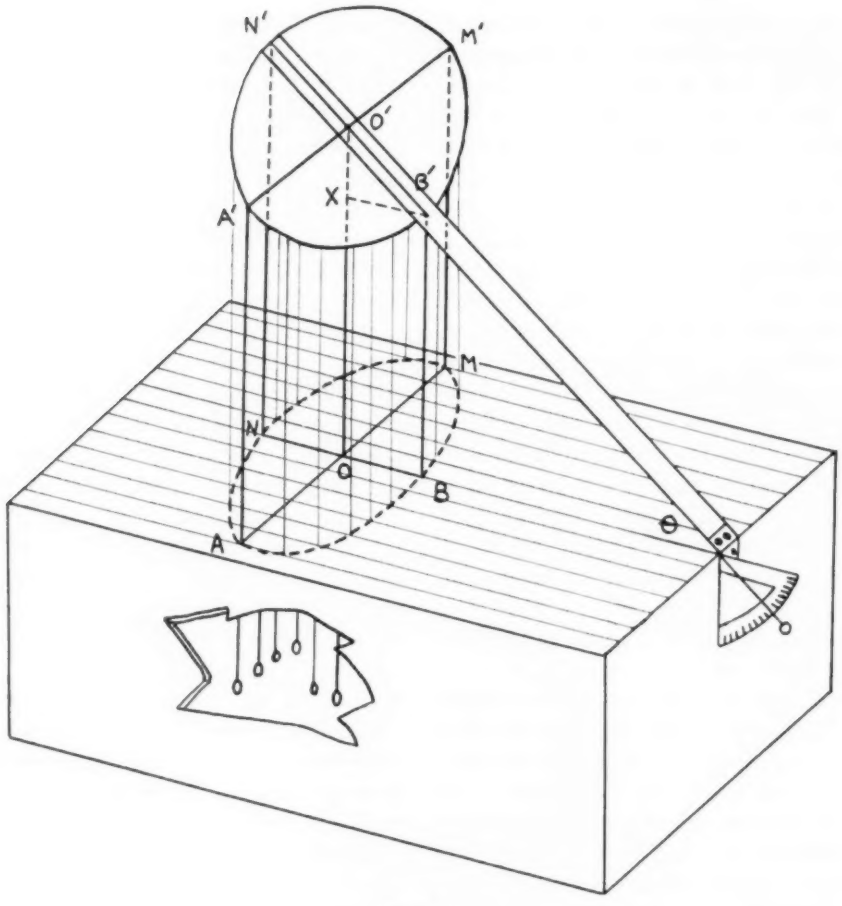


FIG. 6

it was not until the device suggested in this article was used that the students were really satisfied with the development.

THE MATHEMATICS LABORATORY
Monroe High School
St. Paul, Minnesota

[1] David Eugene Smith, William David Reeve, and Edward Longworth Morss, *Essentials of Trigonometry* (Boston: Ginn and Co., 1943) pp. 50–51.

News Notes

(*Continued from page 66*)

with its views of Long Island Sound and the Thames River and its beautiful arboretum and lake, provided an ideal setting for the Institute. Members readily availed themselves of the facilities of the unusual Ocean Beach Park nearby, visited the Lyman Allyn Museum, or explored points of historic interest in New London and vicinity. A picnic at Buck Lodge, a boat trip to Orienta Point, Long Island, and a tour of the United States Coast Guard Academy at New London were among the organized recreational features.

The members of the General Committee which planned and directed the Institute were Charles H. Mergendahl, Chairman; Jackson B. Adkins, Program; George T. Bishop, Exhibits; Caroline Caruso, Recreation; Robert C. Lawton, Housing; Christina S. Little, Laboratory; and Katharine E. O'Brien, Publicity.

KATHERINE E. O'BRIEN
Deering High School
Portland, Maine

The University of Michigan will sponsor a one day **conference** on "General Mathematics in the High School" on Saturday, January 12,

(*Continued on page 76*)

NOTES ON THE HISTORY OF MATHEMATICS

Edited by VERA SANFORD
State Teachers College, Oneonta, New York

The Decimal Point

IN THE earliest description of decimal fractions, Simon Stevin[1] used a symbolism that had certain advantages, although it appears to be clumsy. A number enclosed in a circle indicated the order of the fraction,—① meant the first order or tenths, ② meant the second order or hundredths and so forth. Whole numbers had the symbol ○. These symbols might be written in different ways. Each digit in a fraction might be followed by a symbol as 8① 5② 6③. In computation, the symbols might be written at the top of the proper columns, as in this example

①	②	③
8	5	7
5	0	7
1	3	6	3

When this was inconvenient, the symbols were written below the digits to which they referred or else the highest order of the fraction was expressed by a single circled number at the extreme right. Thus in multiplying .000378 by .54, Stevin arranged the computation in this way:

		④	⑤	⑥	
		3	7	8	
			5	4	②
	1	5	1	2	
1	8	9	0		
2	0	4	1	2	
④	⑤	⑥	⑦	⑧	

[1] Simon Stevin, two versions, *La Thiende*, in Flemish, published in Leyden, *La Disme*, in French, published in Paris. Both dated 1585. A translation is given in the *Source Book in Mathematics*, edited by D. E. Smith.

The advantage of this notation was simply that the highest order of fractions in the product of two numbers was the sum of the highest orders in multiplicand and multiplier. A similar rule applied to the quotient.

Stevin's notation was the basis for several variants. Instead of using enclosing circles, Norton who published a translation of Stevin's work into English in 1608 used parentheses. Beyer (1603), a German, used Roman numerals instead of Hindu Arabic numerals. Napier (1617) sometimes used one, two, three or four oblique lines to indicate the first, second, third or fourth order of fractions. During the seventeenth century, however, this place value notation gave way to the use of a single sign to separate a whole number from a decimal fraction.

An early use of a symbol as a *separatrix* occurs in Clavius's work on the Astrolabe (1593) where he gives the number 46.5 the meaning 46 5/10. Napier (1617) recommended the use of a "period or comma" for this purpose and in his work used both. To quote Cajori,[2] "Napier vacillated between the period and the comma; mathematicans have been vacillating in this matter ever since."

There were other symbols in use—a horizontal line drawn under the numerals in the fractional part of a number, these being printed in smaller type; the use of a vertical line to separate the two parts of a number, the decimal fraction being underscored thus 2.5 appeared as 2|5;

(*Continued on page 73*)

[2] Florian Cajori, *History of Mathematical Notations*, Vol. I (Chicago, 1928) p. 324.

MATHEMATICAL RECREATIONS

Edited by AARON BAKST
135-12 77th Avenue, Flushing 67, N. Y.

Long division may be so tedious that it stretches longer than it actually is. . . . The fundamental multiplication (and division) facts may be so distasteful that the mere thought about them drives a teacher to distraction. . . . And who has ever heard about reasoning in arithmetic? . . . There is enough trouble and worry in getting the pupils to do the simple exercises in the fundamental operations. . . .

No. . . . The space which is allotted to the department of MATHEMATICAL RECREATIONS is not being thrown open for an exchange of complaints. Perhaps the fault with some unsatisfactory results of the teaching of operations with numbers in arithmetic lies with the manner these operations are taught. Perhaps some "dressing up" (and at the same time, a "dressing down" of those who write the myriads of excercises) might be a desirable remedy. . . . Well, here is a series of problems (the readers should note carefully that we use the term "problems" and not "exercises." A problem is not an exercise. In the case of a problem we must discover the method of the attack for the solution, while in the case of an exercise we follow a definitely established routine) which might be helpful. We shall dress-up one such problem in order to impart to it some elements of human touch and humor.

At a meeting of a Board of Education of (well, we might perhaps play safe and omit the name of the school system) the Superintendent of Schools submitted his yearly budget. The Board took one look at the total amount of money required and decided to effect some drastic economies. "Cut the total to one-third," the Board told the Superintendent. "By the way, how much will that be?" "Well," replied the Superintendent, "this will require some division. I think, I better take it to my arithmetic teacher and she will perform the division correctly. You know, we do not want any mistakes. In my younger years I was a social studies teacher and a football coach. I am a bit rusty on my arithmetic."

One of the members of the Board of Education was an accountant. "Look here," he told the Superintendent, "the digit on the extreme left of your total is 7. Take this digit and place it on the extreme right, and you get exactly one third of your total."

"But you will have to check the result by multiplying it by 3," replied the Superintendent, "I am not sure I can rely on my ability to do this correctly either. No, I think I will ask my arithmetic teacher to-morrow."

What was the number which expressed the Superintendent's total?

We know that the digit on the extreme left was 7. Then the number was $7abcd \cdots n$, where $a, b, c, d, \cdots, n$ are digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. The division by 3 is then indicated by the relation

$$7abcd \cdots n \div 3 = abcd \cdots n7.$$

When we divide 7 by 3 we obtain a quotient 2 and a remainder 1. Thus, $a=2$. With the remainder 1 and the number 2 we have 12. Dividing 12 by 3 we obtain a quotient 4. Thus, $b=4$. Dividing 4 by 3 we obtain a quotient 1 and a remainder 1. Thus, $c=1$. With the remainder 1 and the number 1 we have 11. Dividing 11 by 3 we obtain a quotient 3 and a remainder 2. Thus, $d=3$. With the remainder 2 and the number 3 we

have 23. Dividing 23 by 3 we obtain a quotient 7 and a remainder 2. This process is carried out until we finally obtain a quotient 7 and a remainder 0.

Below is reproduced the entire process of division.

The consecutive dividends	The remainder	The quotient	The digits of the number
7	1	2	7
12	0	4	2
4	1	1	4
11	2	3	1
23	2	7	3
27	0	9	7
9	0	3	9
3	0	1	3
1	1	0	1
10	1	3	0
13	1	4	3
14	2	4	4
24	0	8	4
8	2	2	8
22	1	7	2
17	2	5	7
25	1	8	5
18	0	6	8
6	0	2	6
2	2	0	2
20	2	6	0
26	2	8	6
28	1	9	8
19	1	6	9
16	1	5	6
15	0	5	5
5	2	1	5
21	0	7	1

Thus the Superintendent's number was

7,241,379,310,344,827,586,206,896,551.

Actually, this is the smallest number which will satisfy the conditions of the problem. If we use this number as a period of a decimal fraction we obtain a general expression for this number

$$724137931034482758620689655\overline{1} \cdots$$
$$724137931034482758620689655\overline{1} \cdots$$

It may be pointed out here that this property of dividing a number by 3 if a digit on the extreme left of the number is moved to the extreme right may be extended to numbers which begin with any other digit. However, the sequence of the digits as given for the Superintendent's number remains preserved. Thus, we have

1,034,482,758,620,689,655,172,413,793
2,068,965,517,241,379,310,344,827,586
3,103,448,275,862,068,965,517,241,379
4,137,931,034,482,758,620,689,655,172
5,172,413,793,103,448,275,862,068,965
6,206,896,551,724,137,931,034,482,758
8,275,862,068,965,517,241,379,310,344
9,310,344,827,586,206,896,551,724,137

Let no one be frightened by these very large numbers. A school budget will never reach such a magnitude. But this number can be read. Its starts with eightillions (or octillions). Furthermore, the division processes which were indicated above involve at most the division of a two-place number by a one-place number. Surely, in the seventh grade such division should offer no special difficulty. And the problem itself requires a little "thinking it out." Well, why not have it in arithmetic?

History of Mathematics

(*Continued from page 71*)

the use of a colon as a separatrix; a raised comma; a raised and inverted comma. Gradually the countries of the continent adopted the comma for this purpose.

In England, the dot gained favor but was raised above the line of the figures to avoid confusion with the period which in higher mathematics indicated multiplication.

In the United States, in the first half of the nineteenth century, there was a diversity of uses—the raised comma, the raised inverted comma, the raised dot, and the period. There seems to be no evidence that anyone was particularly concerned about the confusion between symbols for decimal fractions and for multiplication. Popular usage adopted the period for a separatrix for decimal fractions. Accordingly about 1880, the raised dot was used to indicate multiplication.

Thus it happens that the number we in the United States write as 3.1416 appears as 3,1416 in France and as $3 \cdot 1416$ in England. The product we write as $2 \cdot 5$ is written as 2.5 in England.

BOOK SECTION

Edited by JOSEPH STIPANOWICH
Western Illinois State College, Macomb, Illinois

BOOKS RECEIVED

ELEMENTARY

Discovering Arithmetic, Book 1, Teachers Edition, by Catherine Stern. Paper, 128 pages, 1951. Houghton Mifflin Company, 2 Park Street, Boston, Mass. $1.60.

HIGH SCHOOL

1. General Mathematics

Everyday General Mathematics, Book Two, by William Betz, A. Brown Miller, F. Brooks Miller, Elizabeth B. Mitchell and H. Carlisle Taylor. Cloth, ix+438 pages, 1951. Ginn and Company, Statler Building, Boston 17, Mass. $2.60.

2. Special Courses in Mathematics

The Arithmetic of Better Business, by Frank J. McMackin, John A. Marsh, and Charles E. Baten. Cloth, viii+389 pages, 1951. Ginn and Company, Statler Building, Boston 17, Mass. $2.48.

COLLEGE

1. College Algebra

Algebra College Course, 2nd Ed., by Raymond W. Brink, University of Minnesota. Cloth, xiii+378 pages, 1951. Appleton-Century-Crofts, Inc., 35 West 32nd Street, New York 1, N. Y. $3.25.

2. Calculus and Analytic Geometry

Calculus and Analytic Geometry, by George B. Thomas, Jr., Massachusetts Institute of Technology. Cloth, 485 pages, 1951. Addison-Wesley Press, Inc., Kendall Square, Cambridge 42, Mass. $6.00.

Differential and Integral Calculus, by Edmund Landau, Göttingen University. Cloth, 366 pages, 1951. Chelsea Publishing Company, 231 W. 29th St., New York 1, N. Y. $5.00.

3. Advanced Mathematics

Principles of Mathematical Logic, by D. Hilbert and W. Ackermann, translated from the German by L. Hammond, G. Leckie, and F. Steinhardt. Cloth, xii+172 pages, 1950. Chelsea Publishing Company, 231 W. 29th St., New York 1, N. Y. $3.50.

Descriptive Geometry—A Pictorial Approach, by Harold Bartlett Howe, Rensselaer Polytechnic Institute. Cloth, x+332 pages, 1951. Ronald Press Company, 15 E. 26th Street, New York 10, N. Y. $4.00.

Foundations of Analysis, by Edmund Landau, Göttingen University, translated by F. Steinhardt. Cloth, xiv+134 pages, 1951. Chelsea Publishing Company, 231 W. 29th St., New York 1, N. Y. $3.25.

Introduction to Number Theory, by Trygve Nagell, University of Uppsala. Cloth, 309 pages, 1951. John Wiley and Sons, Inc., 440 Fourth Ave., New York 16, N. Y. $5.00.

Fourier Series, by Werner Rogosinski. Cloth, vi+176 pages, 1950. Chelsea Publishing Company, 231 W. 29th St., New York 1, N. Y. $2.50.

Introduction to Modern Algebra and Matrix Theory, by O. Schreier and E. Sperner. Cloth, viii+378 pages, 1951. Chelsea Publishing Co., 231 W. 29th St., New York 1, N. Y. $4.95.

TEACHING OF MATHEMATICS

How Big? How Many? by Gladys Risden. Cloth, 248 pages, 1951. The Christopher Publishing House, 1140 Columbus Ave., Boston 20, Mass. $3.50.

Teaching the Meanings of Arithmetic, by C. Newton Stokes, Temple University. Cloth, xi+531 pages, 1951. Appleton-Century-Crofts, Inc., 35 West 32nd St., New York 1, N. Y. $4.50.

MISCELLANEOUS

High-Speed Mathematics, by Lester Meyers. Cloth, xxi+576 pages, 1947. D. Van Nostrand, Inc., 250 Fourth Avenue, New York 3, N. Y. $3.75.

New Visual Education Techniques, by Alfred Porter. Paper, iii+79 pages, 1951. Burgess Publishing Co., 426 S. Sixth St., Minneapolis 15, Minn. $1.75.

REVIEWS

Algebra II: Meaning and Mastery, Daniel W. Snader. Philadelphia, The John C. Winston Co., 1950. 500 pp. $2.60.

The chief aim of the author, as stated in the preface, is to provide a text suited to a program in which individual progress is emphasized. To achieve this end, carefully graded material has been presented in such a way that each student may discover for himself the fundamental rules and laws of algebra and find sufficient drill material for the fixation of these principles.

The author has included a wealth of examples in all the topics usually found in a second course algebra text, including in the

first 100 pages a review of elementary algebra. Especially well written is the section on the slide rule, which, in addition to the usual operations, is extended to cover the solution of trigonometric problems. Graphic analysis of problem solving, an extensive unit on trigonometry including the oblique triangle, and a brief presentation of determinants are indicative of the comprehensive scope of this text. Periodic inventory tests are found in unit one, with chapter reviews placed at the ends of the nine subsequent units. Optional material is starred for the gifted student.

The author seems to have achieved well his aim for a text to be used in those courses geared to the individual progress of the student, while at the same time his text should be more than adequate for the conventional type of program.—CLARENCE W. HACKNEY, Western State High School, Western Michigan College of Education, Kalamazoo, Michigan.

How to Teach Arithmetic, Harry Grove Wheat. Evanston, Illinois, Row, Peterson and Company, 1951. x+438 pp., $3.00.

As the title suggests, this book is written for teachers of arithmetic and for those prepared to teach. Beginning with number names and number signs, the development passes through the study of groups (counting, comparing and analyzing); the addition and subtraction combinations; the idea, extension and use of ten; parts; fractions; measures; related numbers; and applications.

Some features include showing and discussing the interrelations between the four fundamental processes, the author's formula for teaching thinking, and points for the teacher's self improvement, concluding with the story of measure, weights, numeration, and the primitive use of numbers (the latter having value for historical reflection). A few pages show how number thinking is closely interwoven in all of the affairs of the lives of people today.

The book should prove useful to teachers in securing a point of view and an understanding of the arithmetic that they teach.—JOSEPH J. URBANCEK, Chicago Teachers College, Chicago, Illinois.

Teaching the New Arithmetic (New Second Edition), Guy M. Wilson. New York, McGraw-Hill Book Company, 1951. xiv+483 pp. $4.50.

This new second edition presents evidence from a number of recent studies to support the five fundamental principles emphasized in the first edition which was published in 1939. It aims to develop a more rational view of arithmetic as a functional service tool in the lives of children and adults. This is done through emphasis on the main purposes of arithmetic and the limitations that need to be observed. The point is made and stressed that schools should develop arithmetic somewhat beyond the present actual needs of children yet not beyond the needs of adults.

The book is divided into four parts as follows: Part 1. Basic Viewpoints on Arithmetic; Part 2. Processes and Methods; Part 3. Functional Problem Units; and Part 4. Other Considerations for Teachers of Mathematics.

Individuals interested in the teaching of arithmetic will find this book thought-provoking. They may not agree one hundred per cent with the viewpoint of the authors on certain aspects of the teaching of arithmetic yet, by having read the book, they will give greater consideration to that which they teach.

The plea is made for 100% accuracy on the fundamental or essential material. Just what this essential material is, is discussed in Part 2. Chapters 20 and 23 discuss appreciation units and should be read by all teachers of mathematics who hope to have as one of the outcomes of the study of arithmetic some appreciation for it.

Part 3 deals with the subject of problem solving and advocates that Functional Problem Units, developed from situations found in the school community, be employed. A rather extensive list of suggestive units is included, yet no example of such a unit as developed in detail by teacher and pupils is given. In an attempt to show that Functional Problem Units have not been developed in arithmetic textbooks, the author submits problems from many texts, yet, with but one exception the books from which these problems were taken were published previous to the year 1941. In the opinion of the reviewer the material would carry more weight if it had been selected from more than just the one textbook written in the past ten years. In this connection, I quote, page 404, "In further support of the foregoing, 12 problems are noted herewith from two recent arithmetic series, copyrighted respectively in 1926 and 1927." So many arithmetic series have been published since 1927 that it is doubtful if teachers of arithmetic consider arithmetic texts, copyrighted in 1926 and 1927 as recent.

Part 4 concerns remedial arithmetic, a defensible testing program, the selection of a textbook in arithmetic, and further progress in arithmetic.

Appendix A contains 56 selected references of basic studies, twelve of which have been made since 1939, the copyright date of the first edition of *Teaching the New Arithmetic*.—L. H. WHITCRAFT, Ball State Teachers College, Muncie, Indiana.

Secondary Mathematics: A Functional Approach For Teachers, Howard Fehr. Boston, D. C. Heath and Company, 1951. xi+431 pp. $4.25.

This text, "intended to broaden and deepen the teacher's knowledge of the mathematics that he teaches," seems to have something to offer for every adult interested in mathematics, but especially the mathematics teacher.

Dr. Fehr has done an outstanding job of selecting material from advanced mathematics (through elementary calculus) and using them

to enlarge and clarify the topics of secondary mathematics. His writing is clear, thought provoking (filled with stimulating questions), and informative; it suggests a new approach to our old problems. There are many historical notes, detailed references, and practical bits of pedagogy.

The material for the text has the advantage of having been presented several times by the author in lectures before college seniors and graduate students. Of particular interest to the reviewer was the discussion on congruence, curve tracing, loci and the conic section, series, limit, the number concept, complex exponents and roots, integration and differentation, hyperbolic functions, and the three famous problems of antiquity.

Most of the material can be modified for use at the secondary level. Hence, this book is a "must" for secondary mathematics teachers and those interested in the teaching of secondary mathematics.—JOSEPH STIPANOWICH, Western Illinois State College, Macomb, Illinois.

The Teaching of Calculus in Schools, Mathematical Association (England), London, G. Bell & Sons, Ltd., 1951. 76 pp., 4s.6d.

This is a report of a subcommittee of the English Mathematical Association, which is designed to aid the calculus teacher to make the various topics in calculus more meaningful to the beginner.

It begins by suggesting preparatory work in graphs, tangents, and speeds as points of departure in beginning the study of calculus. It is suggested that such topics as limits be described and illustrated rather than formally defined. The different types of notation are discussed along with arguments for and against each type. Other chapters deal with suggestions for teaching such topics as applications of derivatives, function of a function, circular and inverse functions, integration, logarithmic and exponential functions, series, differential geometry and sign-conventions, and differential equations. The report closes with some interesting items from the history of the development of calculus.

The whole report is liberally supplied with suggestions and illustrations which would be of incalculable value to a person beginning to teach calculus. Teachers of experience also will find many things of value in adding interest to certain topics, or in making calculus more meaningful.—CLEON C. RICHTMEYER, Central Michigan College, Mount Pleasant, Michigan.

The Main Stream of Mathematics, Edna E. Kramer. New York, Oxford Press, 1951. xii+309 pp., $5.00.

The Main Stream of Mathematics is a fascinating history of mathematics. The style of writing is delightfully smooth and easy to read. Any mathematical "lingo" is explained so that the layman as well as the mathematician may read and enjoy the story that unfolds. The book begins with the arithmetic of antiquity and continues to the mathematics of atomic energy. Appropriately the volume ends with an interesting discussion of "infinity."

The mathematics teacher will find it a refreshing experience to see how Miss Kramer "teaches" some particular ideas. The teacher may find just the interesting anecdote needed to make a topic he is teaching catch the interest of the pupils. Each chapter is a little history in itself. For instance, Chapter Ten, "A World in Flux," starts with the work of Isaac Newton and develops the ideas of differential and integral calculus. Any mathematics teacher who has this book in his library will use it for that pupil who asks, "What is calculus about?" or "Is mathematics really useful?"—ELIZABETH J. ROUDEBUSH, Seattle Public Schools, Seattle Washington.

News Notes

(*Continued from page 70*)

1952. The meeting will begin at 9:00 A.M. and will be held in the Rackham Building on the Ann Arbor campus.

Knox College announces that it will hold the **1952 Science Prize Competition in Mathematics** on Saturday, March 8, at 10:00 A.M. on the Knox College Campus at Galesburg, Illinois and at 1:00 P.M. at the Chicago Bar Association, 29 S. LaSalle Street, Chicago, Illinois. Two Mabel Heren Prize Scholarships in Mathematics ($1000 and $450) will be offered on the basis of an examination which is briefly described as follows: The examination in mathematics, while presupposing a knowledge of algebra and plane geometry, is primarily designed to measure ability and understanding of mathematics. An applicant competing in mathematics may not compete in another field of science. Only those students are eligible who rank in the highest quarter of their high school class. Each high school may enter three contestants in the field of mathematics. The 1951 prize winners were James M. Goodwin of the Bloom Township High School, Chicago Heights, Illinois and Walter F. Bishop, Tampico Township High School, Tampico, Illinois. Professor Rothwell Stephens, Chairman of the Mathematics Department, states that the announcement in the January, 1951 issue of THE MATHEMATICS TEACHER "contributed no small part to the success of the 1951 contest."

The Committee on Contests and Awards of the Metropolitan New York Section of the Mathematical Association of America announces that the **1952 Mathematics Contest** will be held on Thursday morning, May 1. Participants in former contests have been students in high schools in the Metropolitan New York area. This year it will be possible for any group of schools, such as a city-wide or county-wide group, to organize and use the questions of

the contest provided the contest is held on May 1, 1952. The cost to the schools will be approximately the cost of printing the questions, plus postage and mailing expense. Any school is eligible to register with the Committee and take the contest on the regular date. Last year more than 5,000 students participated. The Bronze Cup of the Association was awarded to Brooklyn Technical High School for a score of 402 points out of a possible 450 points. Certificates of Merit were awarded to the highest ranking students. Those from New York counties were as follows: Bronx County, Lewis Iscol of the Bronx High School of Science; Dutchess County, E. J. Stolz of St. Joseph's Normal Institute; Kings County, P. H. Monsky of the Brooklyn Technical High School; Nassau County, R. A. Howard of the Lynbrook High School; New York County, Alicia E. Larde of the Marymount School; Queens County, R. J. Sibner of the Forest Hills High School; and Westchester County, J. S. Lew of the Mamaroneck Senior High School. Highest ranking students in counties of New Jersey were: Bergen County, J. R. Hastings of the Rutherford High School; Essex County, J. A. Wolf of the College High School of Montclair State Teachers College; Morris County, G. L. Smith of the Chatham High School; and Passaic County, T. E. Douglass of Saint Bonaventure's High School. Honor Keys were awarded to seven students who ranked first in their respective schools for both the 1950 and 1951 contests as follows: Anna Chernovitz of the Washington Irving High School in New York City, P. G. Federbush of the Weequahic High School in Newark, J. S. Lew of Mamaronek, N. Y. Senior High School, Lois B. Mitchell of the Davis High School in Mount Vernon, N. Y., P. H. Monsky of Brooklyn Technical High School, G. L. Smith of the Chatham, N. J. High School, Stephen Weingram of the De Witt Clinton High School in the Bronx. Any high school may obtain information about the contest as well as copies of the 1951 contest questions by writing to the chairman of the Contest Committee, Professor W. H. Fagerstrom, The City College of the City of New York, New York 31, N. Y.

The Duke University Mathematics Institute will be held from **August 5–15, 1952.** The temporary program includes general meetings on history, philosophy, and psychology led by Professor Phillip Jones of the University of Michigan and Professor J. W. Cell of North Carolina State College. Leaders in nine study groups will be L. J. Adams, Allene Archer, Ida Mae Bernhard, Nanette Blackiston, Frances Johnson, Ella Marth, Mary A. Potter, Veryl Schult, and L. G. Woodby. The study topics are: Laboratory Mathematics, Slow Student in Mathematics, Enrichment Materials, Mathematics in Industry, Use of Mathematics Instruments, General Mathematics, Aids in the Study of Algebra, and Basic Principles of Mathematics. For copies of the complete program write to Professor W. W. Rankin, Director, Mathematics Institute, Duke University, Durham, North Carolina.

The National Teacher Examinations, prepared and administered annually by Educational Testing Service, will be given at 200 testing centers throughout the United States on Saturday, February 16, 1952. Application forms and a Bulletin of Information describing registration procedure and containing sample test questions may be obtained from college officials, school superintendents, or directly from the National Teacher Examinations, Educational Testing Service, P.O. Box 592, Princeton, N. J. Completed applications, accompanied by proper examination fees, will be accepted by the ETS office if received before January 18, 1952.

Miss Ina Emma Holroyd, long time teacher of mathematics at Kansas State College and active member of the National Council of Teachers of Mathematics, died in Manhattan, Kansas on October 1, 1951. She was founder of the *Bulletin* of the Kansas Association of Teachers of Mathematics and its editor for eighteen years and served as member of the Board of Directors of the National Council from 1941–3.

Professor J. B. Rosenbach, Head of the Department of Mathematics at Carnegie Institute of Technology, died on November 6, 1951. He held the distinction of being one of the ablest teachers on the Carnegie campus and was a close adviser of both day and evening students. He made important contributions to the Carnegie Plan for Professional Education and served as secretary of the Basic Course Committee for many years. He is the co-author of several texts in college algebra and trigonometry, of "Selected Bibliography of Engineering Subjects, Section I on Mathematics, Physics and Mechanics," and author of "Achievement Tests in Mathematics" and the "Mathematical Section of the Pre-engineering Inventory." The mathematics library at Carnegie has been named the Joseph Bernhardt Rosenbach Mathematics Library in honor of Professor Rosenbach.

Professor Edwin W. Schreiber of Western Illinois State College, Macomb, Illinois, died at his home on December 17, 1951. He was a charter member of the National Council of Teachers of Mathematics, a member of its Board of Directors from 1928 to 1929 and its second Secretary-Treasurer for the twenty-two years from 1929 to 1951. A more detailed account of his life will appear in an early issue.

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